



Department of Mechanical, Materials and
Manufacturing Engineering

Advanced Model Updating Methods for Generally Damped Second Order Systems

by

Latifa Abdalla Abuazoum, BSc. (A.E.) & MSc. (M.E.)

Thesis submitted to the University of Nottingham for the
degree of Doctor of Philosophy

March 31, 2011

**This thesis is dedicated to my parents' souls, Abdalla
Ahmed Abuazoum and Arousyia Ali Abujrida for
their countless sacrifices.**

ان الله وعد المتقين والصابرين المخرج مما يكرهون والرزق من حيث لا يحتسبون. جعلنا الله واياكم من
الذين لا خوف عليهم ولا هم يحزنون

Abstract

This thesis is mostly about the analysis of second order linear vibrating systems. The main purpose of this study is to extend methods which have previously been developed for either undamped or proportionally damped or classically damped systems to the general case. These methods are commonly used in aerospace industries. Ground vibration testing of aircraft is performed to identify the dynamic behaviour of the structure. New aircraft materials and joining methods - composite materials and/or novel adhesive bonding approaches in place of riveted or welded joints - cause higher levels of damping that have not been seen before in aircraft structure. Any change occurring in an original structure causes associated changes of the dynamic behaviour of the structure.

Analytical finite element analyses and experimental modal testing have become essential tools for engineers. These techniques are used to determine the dynamic characteristics of mechanical structures. In Chapters 3 and 4, structural analysis and modal testing have been carried out on an aircraft-like structure. Modal analysis techniques are used to extract modal data which are identified from a single column of the frequency response matrix. The proposed method is presented for fitting modal peaks one by one. This technique overcomes the difficulty due to the conventional methods which require a series of measured FRFs at different points of excitation.

New methods presented in this thesis are developed and implemented initially for undamped systems in all cases. These ideas are subsequently extended for generally damped linear systems. The equations of motion of second order damped systems are

represented in state space. These methods have been developed based on Lancaster Augmented Matrices (*LAMs*) and *diagonalising structure preserving equivalences* (DSPEs).

In Chapter 5, new methods are developed for computing the derivatives of the non-zeros of the diagonalised system and the derivatives of the diagonalising SPEs with respect to modifications in the system matrices. These methods have provided a new approach to the evaluation and the understanding of eigenvalue and eigenvector derivatives. This approach resolves the quandary where eigenvalue and eigenvector derivatives become undefined when a pair of complex eigenvalues turns into a pair of real eigenvalues or vice-versa. They also have resolved when any one or more of the system matrices is singular. Numerical examples have illustrated the new methods and they have shown that the method results overcome certain difficulties of conventional methods.

In Chapter 6, Möbius transformations are used to address a problem where the mass matrix is singular. Two new transformations are investigated called system spectral transformation \mathbf{Q}_{SSTN} and diagonalising spectral/similarity transformation \mathbf{Q}_{DSTO} . The transformation \mathbf{Q}_{SSTN} maps between matrices of two systems having the same short eigenvectors and their diagonalised system matrices. The transformation \mathbf{Q}_{DSTO} maps between two diagonalising SPE's having identical eigenvalues.

Modal correlation methods are implemented to evaluate and quantify the differences between the output results from these techniques. Different cross orthogonality measures represent a class of methods which are recently performed as modal

correlation for damped systems. In Chapter 7, *cross orthogonality* measures and *mutual orthogonality* measures are developed for undamped systems. These measures are defined in terms of real matrices - the *diagonalising structure preserving equivalences* (DSPEs). New methods are well developed for ill-conditioned system such that they work for all occasions and not only for cases where mass matrix is non-singular. Also a measure of the residuals is introduced which does not demand invertibility of diagonalised system matrices.

Model updating methods are used in order to update models of systems by matching the output results from analytical system models with the experimentally obtained values. In Chapter 8, both cross-orthogonality measures and mutual-orthogonality measures are developed and used in the model updating of generally damped linear systems. Model updating based on the mutual orthogonality measures exhibits monotonic convergence from every starting position. That is to say, the ball of convergence has an infinite radius whereas updating procedures based on comparing eigenvectors exhibit a finite ball of convergence.

Craig Bampton transformations are one of component methods which are used to reduce and decouple large structure systems. In Chapter 9 Craig Bampton transformations are developed for undamped systems and extended for damped second order systems in state space. Craig Bampton transformations are generalised and presented in SPEs forms. The two parts of the Craig Bampton transformations are extended in the full sizes of the substructure. The extended Craig Bampton transformations are modified to format each block of transformed substructure matrices as *LAMs* matrices format.

This thesis generalises and develops the methods mentioned above and illustrates these concepts with an experimental modal test and some examples. The thesis also contains brief information about basic vibration properties of general linear structures and literature review relevant to this project.

List of Publications

1. Abuazoum, L. and S.D. Garvey. Eigenvalue and Eigenvector Derivatives from the Perspective of Diagonalising Structure-Preserving. in Proceedings of XXVI International Modal Analysis Conference. 2008. Orlando, Florida, USA.
2. Abuazoum, L.A. and S.D. Garvey, Eigenvalue and eigenvector derivatives of second-order systems using structure-preserving equivalences. Journal of Sound and Vibration, 2009. **323**: p. 1061-1076.
3. Abuazoum, L.A. Spectral Transformations (St's) for Solving ILL-Conditioned Second Order Damped System. in The 3rd Symposium Committee for Libyan Students in UK Universities. 2010. Sheffield Hallam University. 12th June 2010.
4. Abuazoum, L.A. and S.D. Garvey. Modal Correlation for Generally Damped Linear Systems: Basic Principles and Application to Model Updating. The 1st Libyan Engineering Society Meeting - UK Universities. 2010. University of Nottingham. 17th July 2010. Poster.
5. Abuazoum, L.A. and S.D. Garvey, Modal Correlation for Generally Damped Linear Systems: Part I, Basic Principles and Application to Model Updating. Journal of sound and Vibration (in review), 2011.

Acknowledgements

I would like to express my deep gratitude and my appreciation to Professor Seamus Garvey who has seriously supervised and followed up the progress of this research step by step through my PhD journey. For his guidance, giving me all the support, encouragement, venture of letting me think, understand and come to conclusions independently, and for his invaluable expert advice, without which this research would have stumbled.

I would like to thank the University of Nottingham, Al-Fateh University, and Libyan Arab Airlines, all my previous teachers, researchers, colleagues and friends who have assisted and supported me at every stage of my education.

My special deep gratitude and appreciation I would like to dedicate to my parents who had been a lighted candle in my way, for their unwavering support and encouragement through my whole life. I would also like to thank my brothers and sisters and the rest of my family and friends for their help and advice.

List of Contents

Abstract	III
List of Publications	VII
Acknowledgements	VIII
List of Contents	IX
List of Figures	XX
List of Tables.....	XXIV
Abbreviations	XXV
Nomenclature	XXVII
General Notation	XXX
CHAPTER 1. Introduction	1
1.1 Motivation of ground vibration testing (GVT)	1
1.2 System models for vibrating structures	2
1.3 Modal Analysis	3
1.4 Modal Testing	4
1.4.1 Modal testing and its applications	4
1.5 Experimental modal testing setup	5
1.6 Non-proportional and non-classical damping and their measures	9
1.6.1 Real value transformations SPEs	10
1.7 Extracting normal modes from generally damped structures	10
1.8 The Proposed Work	11

1.8.1	Motivation of the proposed work	11
1.8.2	The focus of the work.....	12
1.9	Outline of Thesis.....	14
CHAPTER 2.	Literature Reviews	16
2.1	Basic vibration properties and model structures	17
2.1.1	Classes of vibrating systems	17
2.1.2	Structure modelling	18
2.1.3	System model classes: type and distribution.....	19
2.1.4	Categories of vibration analysis	19
2.1.5	The structure of models of vibrating systems	20
2.2	Ground Vibration Testing (GVT) of Aircraft	21
2.2.1	Validation of structural dynamic models	22
2.2.2	Verification of analytical models	23
2.2.3	Updating analytical models and aircraft certification	23
2.2.4	Identification of structural damage	23
2.2.5	Ground vibration testing methods	24
2.3	Definitions of eigenvalues and eigenvectors	24
2.3.1	Eigenvalues and eigenvectors for undamped systems	24
2.3.2	Eigenvalues and eigenvectors for damped systems	26
2.4	Frequency Response Function	32
2.4.1	Frequency domain calculation using linear models	32
2.4.2	Time domain simulation using linear models	33
2.5	Coordinate transformation	37
2.6	Structure Preserving Equivalences (SPEs)	39
2.6.1	Diagonalising Structure Preserving Equivalences (DSPEs)	42

2.7	Model Reduction and Mode Expansion Methods.....	44
2.7.1	Static Reduction Methods	45
2.7.2	System Equivalent Reduction Expansion Process	47
2.7.3	Improved Reduction System	49
2.7.4	Dynamic Reduction Method	51
2.7.5	Iterative of IRS (IIRS).....	52
2.7.6	Model reduction for damped systems	54
2.7.7	Craig-Bampton Reduction Method	55
2.8	Force appropriation.....	58
2.8.1	Force appropriation methods.....	58
2.8.2	Square FRF matrix method	60
2.8.3	Rectangular FRF matrix method	61
2.9	Modal Correlation Methods.....	62
2.9.1	The Modal Assurance Criterion (MAC)	63
2.9.2	Cross Orthogonality Checks (XOC)	67
2.9.3	Modal correlation for damped system.....	69
2.10	Model Updating	70
2.10.1	Correction of System Matrices using Measured Modal Matrix..	71
2.10.2	Computational Model Updating.....	75
2.10.3	Calculating the Sensitivity Matrix Analytically.....	78
2.10.4	Ball of convergence for model updating	79
2.11	Conclusion	79
CHAPTER 3.	3-D Finite Element Modelling to develop dynamic models	81
3.1	Derivation of finite element formulation	82
3.2	Structural problems analysis	84

3.2.1	Two-dimensional problems.....	85
3.2.2	Three- dimensional problems.....	85
3.3	FE modelling 3-D	85
3.3.1	Shape functions	86
3.3.2	Natural coordinate system.....	88
3.3.3	The stress-strain relationship.....	90
3.3.4	Element stiffness matrix from potential energy	92
3.3.5	Element stiffness matrix Assembly.....	94
3.4	Structural analysis dimensions and properties.....	94
3.4.1	Meshing the structure	95
3.4.2	Material Properties and an appropriate element ordering	96
3.4.3	Derive the element matrices	99
3.4.4	Maximum front size	100
3.4.5	Element matrices	102
3.4.6	Assembling the element matrices.....	102
3.4.7	Eigenvalues and eigenvectors	103
3.5	Modes of Vibration.....	103
3.6	Conclusion	104
CHAPTER 4.	Modal Testing	108
4.1	General Outline of Modal Testing	109
4.1.1	Measurement setup.....	110
4.1.2	Excitation Methods	111
4.1.3	Response measurements.....	111
4.1.4	Obtaining Frequency Response Function (FRFs)	112
4.2	Work done on the GARTEUR-like aircraft structure	116

4.3	Calibration of a piezoelectric accelerometer	118
4.3.1	Frequency response function (FRF) of linear system	120
4.3.2	Correction factor C_f	121
4.4	Extracting modal information	124
4.4.1	A single column of the frequency Response- $g(\omega)$	125
4.4.2	Extracting modes given $\{\Lambda, \Gamma, I\}$	127
4.4.3	Iteratively adjusting $\{\Lambda, \Gamma\}$ to minimise vector of residuals Δr	128
4.4.4	Obtaining initial values for the parameters $\{\Lambda, \Gamma\}$	131
4.5	Modes of vibration	134
4.6	Comparison of modal data	135
4.7	Conclusions.....	137
CHAPTER 5.	Eigenvalue and Eigenvector Derivatives	141
5.1	Eigenvalue and eigenvector derivative methods	144
5.2	Eigenvalue and eigenvector derivatives for undamped systems	150
5.3	Rates of change of the diagonalised matrices for undamped systems	153
5.4	Orthogonal transformation matrix - Householder Reflection.....	154
5.5	Rates of change of the diagonalising transformation for undamped system	155
5.6	Eigenvalues and eigenvectors for damped systems	159
5.7	Diagonalising structure-preserving equivalences (DSPEs)	161
5.8	Derivatives of diagonalised damped system.....	165
5.8.1	Eigenvalues and their derivatives from diagonalised systems ..	167
5.9	Derivatives of the diagonalising transformations	168

5.9.1	Eigenvectors and their derivatives from diagonalising transformations.....	174
5.10	Examples.....	175
5.10.1	Example 1: A pair of identical real roots	176
5.10.2	Example 2: Singular Mass matrix	177
5.10.3	Example 3: Undamped non-symmetric system.....	179
5.10.4	Example 4: A Physical system	181
5.11	Conclusions.....	182
CHAPTER 6.	Spectral Transformations (STs)	184
6.1	Eigenvalue problems for general second order systems.....	185
6.2	Möbius Transformations	186
6.3	Decomposing the Möbius transformations	191
6.4	Shift spectral transformation (ST _s).....	192
6.4.1	Shift spectral transformation and diagonalising SPEs	194
6.5	Invert spectral transformation (ST _i)	194
6.5.1	Invert spectral transformation and diagonalising SPEs	195
6.6	Shift spectral transformation (ST _s).....	196
6.6.1	Shift spectral transformation and diagonalising SPEs	197
6.7	Scale spectral transformation (ST _{sc})	197
6.7.1	Scale spectral transformation and diagonalising SPEs	198
6.8	Composing spectral transformations	199
6.9	Example	202
6.10	Conclusion	205
CHAPTER 7.	Modal correlation	207

7.1	Undamped system modes	208
7.1.1	Eigenvalues and eigenvectors	208
7.1.2	Mass and stiffness weighted orthogonality for undamped systems 209	
7.2	Generally damped modes for second-order systems	210
7.2.1	Eigenvalues and eigenvectors for damped systems	210
7.2.2	Structure preserving equivalences (SPEs).....	211
7.3	Modal correlation for undamped systems with complete modal information	213
7.3.1	Orthogonality properties of undamped systems.....	213
7.3.2	Mass and stiffness-weighted cross-orthogonality for undamped systems	215
7.3.3	Mass and Stiffness mutual orthogonality for undamped system 216	
7.3.4	Rate of change of cross orthogonality measures \mathbf{O}_{Moo} and \mathbf{O}_{Koo} for undamped systems	218
7.3.5	Residual measures based on mutual orthogonality for undamped systems	220
7.4	Modal correlation for undamped systems with incomplete modal information	222
7.5	Modal Correlation measures for general second order systems	224
7.5.1	Orthogonality properties of damped systems.....	224
7.5.2	Cross orthogonality measures for general damped systems.....	226
7.5.3	Mutual orthogonality for general damped systems	228

7.5.4	Rate of change of different cross orthogonality measures $\underline{\mathbf{XO}}_{Ma}$, $\underline{\mathbf{XO}}_{Da}$ and $\underline{\mathbf{XO}}_{Ka}$ for damped systems	231
7.5.5	Residual measures based on mutual orthogonality for second order system	234
7.6	Example	238
7.7	Conclusion	241
CHAPTER 8.	Model Updating.....	242
8.1	Model updating	243
8.1.1	Residuals and the sensitivity matrix.....	244
8.2	Model updating for undamped system.....	247
8.2.1	Model updating based on eigenpairs of undamped system.....	248
8.2.2	Model updating based on cross orthogonality measures of undamped systems.....	251
8.2.3	Model updating based on mutual orthogonality measures of undamped systems.....	253
8.3	Model updating of generally damped systems	254
8.3.1	Model updating based on eigenvalues and eigenvectors for general damped systems.....	255
8.3.2	Model Updating based on diagonalising SPEs	260
8.3.3	Model updating based on homogeneous coordinates.....	266
8.3.4	Model updating based on cross orthogonality measures for general damped systems.....	270
8.3.5	Model updating based on mutual orthogonality measures.....	272
8.4	Example-1	273

8.5	Example-2	278
8.6	Conclusion	282
CHAPTER 9.	Craig-Bampton Transformations (CBTs).....	284
9.1	Undamped substructure systems.....	285
9.2	Craig-Bampton transformations for undamped substructure systems	286
9.2.1	Matrix spanning the space orthogonal to \mathbf{S}_s for undamped substructure systems.....	287
9.2.2	Finding the matrix \mathbf{F} for undamped substructure systems	288
9.2.3	Finding the matrix \mathbf{E} for undamped substructure systems	289
9.2.4	Substructure model matrices for undamped systems	291
9.3	Damped substructure	292
9.4	Craig-Bampton transformations for damped second order systems ...	294
9.4.1	Finding the matrix $\underline{\mathbf{F}}$ for damped substructure systems in state space	295
9.4.2	Finding the matrix $\underline{\mathbf{E}}$ for damped substructure systems in state space	296
9.4.3	Substructure model matrices for damped systems	297
9.5	Generalised Craig-Bampton Transformations	298
9.5.1	Craig Bampton transformations in SPEs form.....	299
9.5.2	Craig Bampton transforms $\{\underline{\mathbf{E}}_R, \underline{\mathbf{E}}_L\}$ in the full size of the substructure	301
9.5.3	Craig-Bampton transforms $\underline{\mathbf{F}}_R, \underline{\mathbf{F}}_L$ in the full size of the substructure	304
9.6	Modified Craig Bampton transformations and SPEs.....	305

9.6.1	Modified Craig-Bampton transforms $\{\underline{\mathbf{E}}_R, \underline{\mathbf{E}}_L\}$ in the full size of the substructure	307
9.6.2	Modified Craig-Bampton transforms $\underline{\mathbf{F}}_R, \underline{\mathbf{F}}_L$ in the full size of the substructure	308
9.7	Example	309
9.8	Conclusion	312
CHAPTER 10.	Conclusions and further work	313
10.1	Conclusions.....	313
10.2	Further work	316
10.2.1	Applications on real aircraft structures	317
10.2.2	Excitation and response devices.....	317
APPENDICES	318
Appendix A.	Proof of diagonalising SPEs $\{\underline{\mathbf{T}}_R, \underline{\mathbf{T}}_L\}$	318
Appendix B.	Binomial theorem.....	321
Appendix C.	Proof of two equivalent statements	322
Appendix D.	Calculating eigenvalue, eigenvector and thier derivatives from diagonalised system matrices and DSPEs.....	324
D.1	Calculating pairs of eigenvalues and their derivatives from the diagonalised system matrices and its derivatives.....	327
D.2	Calculating two columns of eigenvectors and their derivatives from DSPEs and its derivatives.....	328
Appendix E.	Extracting the measured DSPEs using Modal filters based on Möbius transformation.....	330

Appendix F. Calculating measured DSPTs using homogenous coordinates based on Möbius transformation	334
Bibliography.....	342

List of Figures

Figure 1-1 : The soft support system	6
Figure 1-2 : Electrodynamic exciters - shakers.....	7
Figure 1-3 : Impact hammer.....	7
Figure 1-4 : Piezoelectric accelerometer.....	8
Figure 1-5 : Block diagram of a linear system.....	9
Figure 2-1 : Structures of dynamic model.....	18
Figure 2-2 : Theoretical vibration analysis	20
Figure 2-3 : Experimental vibration analysis	20
Figure 2-4 : Model structure in 3 stages.....	21
Figure 2-5 : Arbitrary forcing function	36
Figure 2-6 : System coordinates.....	44
Figure 2-7 : Master and slave coordinates	45
Figure 2-8 : Boundary and interior coordinates	56
Figure 3-1 : GARTEUR-like aircraft structure	82
Figure 3-2 : Quadratic hexahedron (Brick) 20-Noded element	86
Figure 3-3 : Reference coordinates for the corner node points of 20-noded element	87
Figure 3-4 : Reference coordinates for middle node points of 20-noded element.....	88
Figure 3-5 : Finite element meshing	96
Figure 3-6 : Global origin point of the structure	98
Figure 3-7 : History of the minimum front size	101
Figure 3-8 : Mode (7) Freq. 13.44 Hz.....	106
Figure 3-9 : Mode (8) Freq. 29.98 Hz.....	106
Figure 3-10 : Mode (9) Freq. 34.76 Hz.....	106

Figure 3-11 : Mode (10) Freq. 38.41 Hz	106
Figure 3-12 : Mode (11) Freq. 48.57 Hz	106
Figure 3-13 : Mode (12) Freq. 51.82 Hz	106
Figure 3-14 : Mode (13) Freq. 56.90 Hz	106
Figure 3-15 : Mode (14) Freq. 60.40 Hz	106
Figure 3-16 : Mode (16) Freq. 88.89 Hz	107
Figure 3-17 : Mode (17) Freq. 107.50 Hz	107
Figure 3-18 : Mode (18) Freq. 117.50 Hz	107
Figure 3-19 : Mode (19) Freq. 136.49 Hz	107
Figure 3-20 : Mode (20) Freq. 147.07 Hz	107
Figure 4-1 : GARTEUR-like aircraft structure	109
Figure 4-2 : Transducers to measure forces and responses	111
Figure 4-3 : Piezoelectric accelerometers	112
Figure 4-4 : FRF and coherence	116
Figure 4-5 : Model structure	117
Figure 4-6 : Location of the accelerometer on airplane structure	117
Figure 4-7 : Equipments used in modal testing.....	118
Figure 4-8 : Mass weighted.....	119
Figure 4-9 : Model of calibration	120
Figure 4-10 : Single degree of freedom system	120
Figure 4-11 : Calibration curve of piezoelectric accelerometer	122
Figure 4-12 : Input and output signals diagram	123
Figure 4-13 : Calibration curves of piezoelectric accelerometer	124
Figure 4-14 : Modal peaks	131
Figure 4-15 : Modal peaks / frequency-bands.....	132

Figure 4-16 : Error vs. Log of damping factor	133
Figure 4-17 : Reconstructed modal peaks	134
Figure 4-18 : Mode (1) Freq. 29.96 Hz	139
Figure 4-19 : Mode (2) Freq. 35.88 Hz	139
Figure 4-20 : Mode (3) Freq. 37.32 Hz	139
Figure 4-21 : Mode (4) Freq. 45.46 Hz	139
Figure 4-22 : Mode (5) Freq. 50.46 Hz	139
Figure 4-23 : Mode (6) Freq. 56.41 Hz	139
Figure 4-24 : Mode (7) Freq. 57.47 Hz	139
Figure 4-25 : Mode (8) Freq. 57.90 Hz	139
Figure 4-26 : Mode (9) Freq. 92.01 Hz	140
Figure 4-27 : Mode (10) Freq. 107.70 Hz	140
Figure 4-28 : Mode (11) Freq. 121.98 Hz	140
Figure 4-29 : Mode (12) Freq. 134.68 Hz	140
Figure 4-30 : Mode (13) Freq. 140.48 Hz	140
Figure 5-1 : Root locus for one pair of roots	177
Figure 5-2 : Physical system; mass, stiffness and damping	181
Figure 6-1 : Angles θ , β and condition of M_N	188
Figure 6-2 : Möbius transformation diagram	190
Figure 6-3 : Spectral transformation diagram and SPEs	200
Figure 6-4 : Angles θ and β with condition number of M_N	204
Figure 6-5 : Contours of condition number of M_N	204
Figure 8-1 : Projection of \mathbf{v}_{Ri} into \mathbf{w}_{Rj}	258
Figure 8-2 : Physical system structure	274
Figure 8-3 : Radius of ball of convergence	281

Figure 9-1 : Substructure boundary and interior DoFs 286

List of Tables

Table 2-1 : Average of the interval h	35
Table 3-1 : Gauss points and coefficients	93
Table 3-2 : Material properties of aluminium	97
Table 3-3 : Minimum front size	101
Table 3-4 : Natural frequencies and computed mode shapes	104
Table 4-1 : Experimental natural frequencies and extracted modes	135
Table 4-2 : Comparison of modal data from different tests	137
Table 5-1 : The diagonalised system and its rate of change	177
Table 5-2 : Diagonalised system and its rates of change	179
Table 5-3 : Diagonalised system and its rates of change	180
Table 5-4 : Diagonalised system and its rates of change	182
Table 6-1 : Decomposition of the Möbius transformation $\lambda \rightarrow \gamma$	191
Table 6-2 : Decomposition of the Möbius transformation $\gamma \rightarrow \lambda$	192

Abbreviations

CBTs	Craig Bampton Transformations
CMS	Component mode synthesis
COMAC	Coordinate Modal Assurance Criterion
COOC	Coordinate Orthogonality Check
DoFs	Degrees of freedom
DSPEs	Diagonalising Structure Preserving Equivalences
DST_i	Invert Diagonalising Spectral Transformation
DST_s	Shift Diagonalising Spectral Transformation
DST_sc	Scale Diagonalising Spectral Transformation
FRAC	Frequency Response Assurance Criterion
FEM	Finite Element Model or Method
FRF	Frequency Response Function
FSMAC	Frequency Scaled Modal Assurance Criterion
GARTEUR	Group for aeronautical research and technology in Europe
GVT	Ground Vibration Test
IIRS	Iterative of Improved Reduction System
IRF	Impulse Response Function
IRS	Improved Reduction System
LAMs	Lancaster Augmented Matrices
MAC	Modal Assurance Criterion
MDOF	Multi Degrees of Freedom
MIMO	Multi Inputs and Multi Outputs
MMIF	Multivariate Mode Indicator Function

MSF	Modal Scale Factor
PMAC	Partial Modal Assurance Criterion
SDOF	Single Degree of Freedom
SEREP	System Equivalent Reduction Expansion Process
SIMO	Single Input and Multi Outputs
SISO	Single Input and Single Output
SMAC	Scaled Modal Assurance Criterion
SPTs	Structure Preserving Transformations
SPEs	Structure Preserving Equivalences
SST	System Spectral Transformation
SST _i	Invert System Spectral Transformation
SST _s	Shift System Spectral Transformation
SST _{sc}	Scale System Spectral Transformation
SVD	Singular Value Decomposition
XOC	Cross Orthogonality Checks

Nomenclature

d_i	The i^{th} entry of \mathbf{D}_D
\mathbf{f}	Vector of forces
$\mathbf{g}(\omega)$	A single column of frequency response
$\{\mathbf{g}_{Li}, \mathbf{g}_{Ri}\}$	Vectors
k_i	The i^{th} entry of \mathbf{K}_D
m	Number of measured modes
m_i	The i^{th} entry of \mathbf{M}_D
n	Number of DoFs
n_i, n_o	Input and output number of DoFs
\mathbf{q}	Vector of displacements
\mathbf{r}	Vector of residuals
t	Time
\mathbf{u}	Input forces at terminal DoFs or unit vector
x, y, z	Cartesian coordinates
\mathbf{y}	Output displacement at terminal DoFs
$\{\alpha_L, \alpha_R\}$	Scalars
$\{\beta_{Li}, \beta_{Ri}\}$	Unknown scalars
λ	Eigenvalue
ω, ω_n	Frequency
$\alpha, \beta, \gamma, \theta$	Angles or variables
ζ, η, ξ	Local coordinates
ζ	Damping ratio

ϕ_R, ϕ_L	Right and left eigenvectors
σ	scalar parameter
θ	Vector of parameters
C_f	Correction factor
\mathbf{D}	Damping matrix
\mathbf{D}_D	Diagonal matrix corresponding to the damping
$\underline{\mathbf{D}}$	LAMs matrix does <i>not</i> contain \mathbf{D}
$\underline{\mathbf{D}}_D$	Diagonal block does <i>not</i> contain \mathbf{D}_D
\mathbf{E}	Second part of Craig-Bampton transformations
$\mathbf{F}_R, \mathbf{F}_L$	Right and left SPEs construction parameters
\mathbf{F}	First part of Craig-Bampton transformations
$\mathbf{G}_R, \mathbf{G}_L$	Right and left SPEs construction parameters
$\mathbf{H}(\omega)$	Frequency response function FRF
\mathbf{I}	Identity matrix
\mathbf{K}	Stiffness matrix
\mathbf{K}_D	Diagonal matrix corresponding to the stiffness
$\underline{\mathbf{K}}$	LAMs does <i>not</i> contain \mathbf{K}
$\underline{\mathbf{K}}_D$	Diagonal block does <i>not</i> contain \mathbf{K}_D
\mathbf{M}	Mass matrix
\mathbf{M}_D	Diagonal matrix corresponding to the mass
$\underline{\mathbf{M}}$	LAMs does <i>not</i> contain \mathbf{M}
$\underline{\mathbf{M}}_D$	Diagonal block does <i>not</i> contain \mathbf{M}_D
N_i	Mode shape function
$\mathbf{O}_{Moo}, \mathbf{O}_{Koo}$	Mass and stiffness mutual orthogonality

Q	Householder reflection matrix
R	Residuals
S	Sensitivity matrix
S_R, S_L	Right and left selection matrices
T	Transformation
T_{cb}	Craig-Bampton transformation matrix
T_R, T_L	Right and left transformations or SPEs matrices
XO_M	Mass weighted cross orthogonality
XO_K	Stiffness weighted cross orthogonality
X_R, X_L	Right and left SPEs construction parameters
Y_R, Y_L	Right and left SPEs construction parameters
Ω, Ψ, Γ	Diagonal matrices
Θ, Ψ	Diagonal matrices corresponding to M_D , and K_D
<u>Ω</u>, <u>Ψ</u>, <u>Γ</u>	Double dimension diagonal matrices
Λ	Eigenvalues matrix
Φ_R, Φ_L	Right and left modal matrices
Φ̇_R, Φ̇_L	Derivatives of modal matrices with respect to scalar parameter

General Notation

Underline notation used to represent double dimension ($2n$)

The dot (.) above a vector denotes derivative with respect to time or parameter.

Superscript “T” denotes transpose.

Superscript notation “*” indicates a complex conjugate.

Superscript notation “+” denotes generalised inverse or pseudo inverse.

Subscripts “R” and “L” represent right and left.

Subscript “a” indicates the analytical system

Subscript “c” indicates complex modes.

Subscripts “ r_1 ” and “ r_2 ” indicate real modes.

Subscript “o” indicates the original system.

Subscript “m” indicates the measured system

Subscript “N” indicates the new system.

Subscript “r” denotes reduced (master) coordinates.

Subscript “e” denotes eliminated (slave) coordinates or element

Subscript “up” indicates updated models.

Matrices are denoted by bold uppercase characters.

Vectors are denoted by bold lower case characters.

Scalars are represented by non-bold italicized characters.

The notation “ $\frac{\partial(.)}{\partial\theta}$ ” denotes partial derivatives.

The notation “ $|\cdot|$ ” denotes Euclidean length of vector.

The notation “ \equiv ” denotes a definition.

CHAPTER 1. Introduction

1.1 Motivation of ground vibration testing (GVT)

The safety of aircraft structures depends at least partly on their dynamic behaviour. These characteristics which are called modal parameters include natural frequencies, damping ratio and mode shapes. The modal parameters are commonly used in the validation of analytical (finite element) structural models and studies of structural vibration [1]. These quantities also can be used to identify any structural damage [2].

Large and complex structures such as aircraft after some time of operation face a great deal of structural damage such as cracks. Such structural damage changes to some extent the modal parameters of the structures. These changes can be noticed in shifts in the resonant frequencies.

Analytical finite element method and experimental modal testing are two techniques commonly used to determine the dynamic characteristics of mechanical structures [3]. Both techniques have played dominant roles in aircraft design, manufacture, research and development of a wide range of aerospace industries. Also, they have become most important implements for engineers to improve the performance, safety and reliability of various structures [2].

Ground vibration testing (GVT) is performed to identify the dynamic behaviour of these structures by causing the structure to shake and then measuring its responses. Resonant frequencies and associated mode-shapes can be extracted from these tests.

In addition, the main purpose of GVT is to obtain measured data of the whole aircraft structure for validating and improving its structural dynamic models [4, 5]. Furthermore, ground vibration testing plays an important role for updating of analytical models in the certification of new aircraft [6]. Verification of analytical models using ground vibration test results is common practice in many aerospace applications [7].

1.2 System models for vibrating structures

The equation of motion governing the vibration of a general damped second order system can be written as

$$\begin{aligned}\mathbf{f}(t) &= \mathbf{S}_L \mathbf{u} \\ \mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{D}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) &= \mathbf{f}(t) \\ \mathbf{y} &= \mathbf{S}_R^T \mathbf{q}(t)\end{aligned}\tag{1.1}$$

where $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ are $(n \times n)$ stiffness, damping and mass matrices, \mathbf{q} is a vector of displacement coordinates, \mathbf{f} is a vector of forces. Both vectors are functions of time and the system has n degrees of freedom. The vector \mathbf{y} represents output displacements and it has n_o entries. The vector \mathbf{u} represents the corresponding input forces and it has n_i entries. \mathbf{S}_R ($n \times n_o$) and \mathbf{S}_L ($n \times n_i$) are right and left matrices relating the terminal variables to the model degrees of freedom.

In order to solve equations (1.1) for the response \mathbf{q} the major obstacle is the coupling between the equations. In subsequent sections, we will discuss coordinate transformations which decouple the equations.

1.3 Modal Analysis

The eigenvalues and the corresponding eigenvectors satisfy the following equations

$$\left(\mathbf{M}\lambda_i^2 + \mathbf{D}\lambda_i + \mathbf{K}\right)\boldsymbol{\phi}_{Ri} = 0 \quad , \quad \boldsymbol{\phi}_{Li}^T \left(\mathbf{M}\lambda_i^2 + \mathbf{D}\lambda_i + \mathbf{K}\right) = 0 \quad (1.2)$$

in which λ_i is the i^{th} eigenvalue and $\{\boldsymbol{\phi}_{Ri}, \boldsymbol{\phi}_{Li}\}$ are the right and left eigenvectors.

$$\boldsymbol{\Phi}_R = [\boldsymbol{\phi}_{R1} \ \boldsymbol{\phi}_{R2} \ \dots \ \boldsymbol{\phi}_{Rn}] \quad , \quad \boldsymbol{\Phi}_L = [\boldsymbol{\phi}_{L1} \ \boldsymbol{\phi}_{L2} \ \dots \ \boldsymbol{\phi}_{Ln}] \quad (1.3)$$

Modal analysis is a process of deriving system responses by transforming the equations of motion into a set of decoupled equations [8]. Any pair of matrices $\{\mathbf{T}_R, \mathbf{T}_L\}$ can be used to define a coordinate transformation according to the following equations.

$$\begin{aligned} \tilde{\mathbf{f}} &= \mathbf{T}_L^T \mathbf{f} \\ \mathbf{T}_L^T \mathbf{M} \mathbf{T}_R \ddot{\tilde{\mathbf{q}}} + \mathbf{T}_L^T \mathbf{D} \mathbf{T}_R \dot{\tilde{\mathbf{q}}} + \mathbf{T}_L^T \mathbf{K} \mathbf{T}_R \tilde{\mathbf{q}} &= \tilde{\mathbf{f}} \\ \mathbf{q} &= \mathbf{T}_R \tilde{\mathbf{q}} \end{aligned} \quad (1.4)$$

where $\{\mathbf{T}_R, \mathbf{T}_L\}$ are the right and left coordinate transformation matrices. If $\{\mathbf{T}_R, \mathbf{T}_L\}$ are invertible, the spectrum of the original system is identical to the spectrum of the new system such that the dynamic models can be written as

$$\mathbf{T}_L^T \mathbf{M} \mathbf{T}_R = \mathbf{M}_N \quad (1.5)$$

$$\mathbf{T}_L^T \mathbf{K} \mathbf{T}_R = \mathbf{K}_N \quad (1.6)$$

$$\mathbf{T}_L^T \mathbf{D} \mathbf{T}_R = \mathbf{D}_N \quad (1.7)$$

It is not usually possible to find invertible matrices, $\{\mathbf{T}_R, \mathbf{T}_L\}$ such that the new system matrices are all diagonal. There are some systems which are proportionally and classically damped. For proportional damping, the damping matrix is a function of the mass and stiffness matrices. In the case of proportional damping, the coordinate transformations $\{\mathbf{T}_R, \mathbf{T}_L\}$ decouple the system matrices.

Classical damping is a more general condition. Caughey and O’Kelly [9] expressed one sufficient criterion to satisfy equations (1.5), (1.6) and (1.7) in which $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ are diagonal. The coordinate transformations decouple the system matrices if and only if one of the following conditions are satisfied [9].

$$\begin{aligned}\mathbf{D}\mathbf{M}^{-1}\mathbf{K} &= \mathbf{K}\mathbf{M}^{-1}\mathbf{D} \\ \mathbf{D}\mathbf{K}^{-1}\mathbf{M} &= \mathbf{M}\mathbf{K}^{-1}\mathbf{D} \\ \mathbf{M}\mathbf{D}^{-1}\mathbf{K} &= \mathbf{K}\mathbf{D}^{-1}\mathbf{M}\end{aligned}\tag{1.8}$$

When $\{\mathbf{T}_R, \mathbf{T}_L\}$ represent diagonalising transformations, the structure will behave as a collection of single degree of freedom (SDOF) systems. The second equation in (1.4) can be written as

$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & m_n \end{pmatrix} \begin{Bmatrix} \ddot{\mathbf{q}}_1 \\ \vdots \\ \ddot{\mathbf{q}}_n \end{Bmatrix} + \begin{pmatrix} d_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & d_n \end{pmatrix} \begin{Bmatrix} \dot{\mathbf{q}}_1 \\ \vdots \\ \dot{\mathbf{q}}_n \end{Bmatrix} + \begin{pmatrix} k_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & k_n \end{pmatrix} \begin{Bmatrix} \mathbf{q}_1 \\ \vdots \\ \mathbf{q}_n \end{Bmatrix} = \begin{Bmatrix} \tilde{\mathbf{f}}_1 \\ \vdots \\ \tilde{\mathbf{f}}_n \end{Bmatrix}\tag{1.9}$$

1.4 Modal Testing

Vibration measurements generally comprise two types of test. The first one is measuring the vibration forces or responses of the structures under study or during operation. The second one involves vibrating the structures or components with known excitation. The latter test, including data acquisition and its subsequent analysis is called modal testing [10].

1.4.1 Modal testing and its applications

Validating the mathematical model used to describe the dynamics of components or structures is usually the major application of modal testing. There are different

applications of modal testing in order to produce the mathematical model of structures or components.

1. The most commonly application is the measurement of vibration properties (natural frequencies, mode shapes and damping factor) of a structure which are used to compare and validate the theoretical model.
2. Using modal testing in order to produce the mathematical model of a component is the next application. Complex structures such as aircraft contain many substructures. The dynamic behaviour for each component is required to include that component into a structural assembly. These identifications and accurate data from that substructure are also required for modifications of the original structure.
3. Force determination is a different application of using modal testing. The dynamic forces causing vibration are required. These forces can be deduced by measuring the response caused by the forces. The measurements of response are combined with the mathematical model of the transfer functions of the structure.

1.5 Experimental modal testing setup

Preparation of the test structure itself is an essential element of modal testing. How to support the structure is the first decision which has to be taken. This determines whether the structure is to be tested in a free condition or grounded. Free support

means that the structure is not attached to ground (rigidly) at any of its coordinates. Modal testing of an aircraft may involve the complete aircraft mounted on a soft support system to simulate free-free condition, resting on its landing gear, specific parts on the complete airplane, or components, such as landing gear doors or control surfaces [11]. Figure 1-1 shows the soft support system.



Figure 1-1 : The soft support system

The main pieces of equipment used in experimental modal testing setup are excitation devices, dynamic response and a spectrum analyser. Figure 1-2 and Figure 1-3 show excitation devices such as electrodynamic vibration exciters and an impact hammer. The excitation devices provide a measurable input force into the test structure. The input force signal is typically measured by means of a force transducer.

The type of analysis to be performed and accuracy requirements of test information will influence the final choice of excitation signal. Transient excitation is usually by means of a force impulse from an impact hammer. This method is suitable for a wide

variety of engineering structures and is not suitable for non-linear systems. Random excitation is from an exciter via a push-rod or stinger. Direct excitation by exciters is the best excitation method for linear systems, giving a best linear approximation of system frequency response function [12].



Figure 1-2 : Electrodynamic exciters - shakers

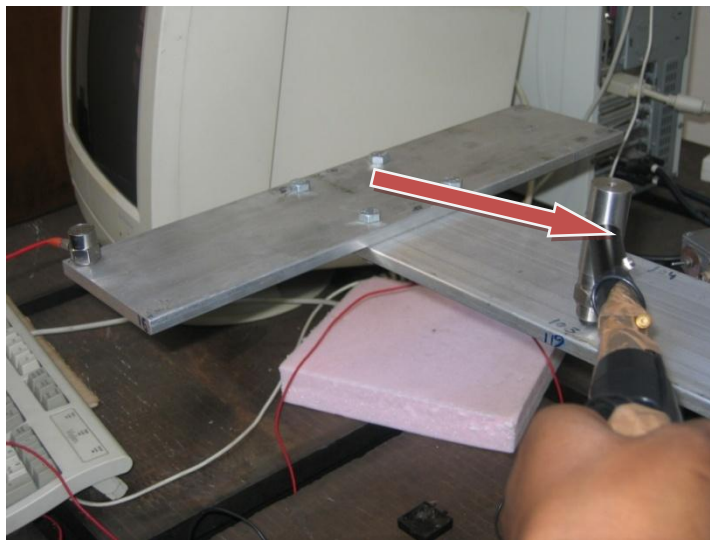


Figure 1-3 : Impact hammer

The dynamic response of an excited test structure is normally measured by one or more piezoelectric accelerometers (Figure 1-4) attached to the structure. The selection of the accelerometers (transducers) is influenced by the sensitivity, bandwidth, linearity and the working range.

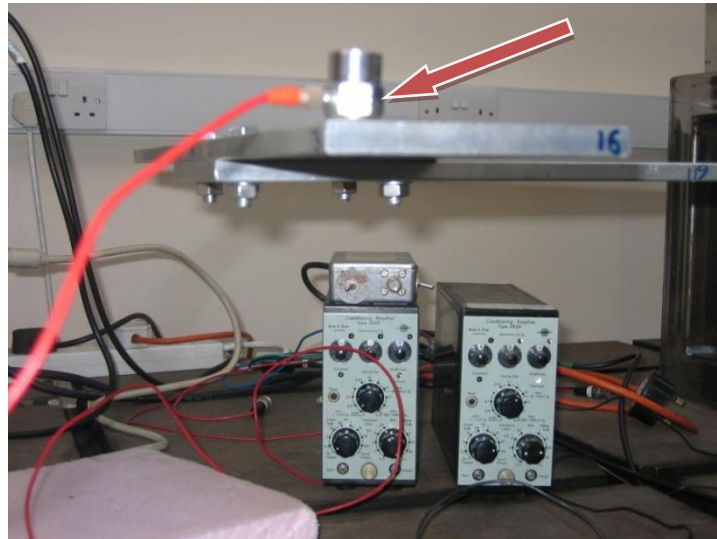


Figure 1-4 : Piezoelectric accelerometer

A spectrum analyser is used to measure the various signals developed by the transducers in order to determine the magnitudes of the excitation forces and responses. The ratio between the output signals and the input signals is called the frequency response function (FRF). Frequency response functions are properties of a linear dynamic system and they do not depend on the type of excitation. For such structures, excitation can be a harmonic, random or transient function of time.

In any linear system, there is a direct linear relationship between the input and output. This relationship, which also holds for random functions, is represented by the block diagram of a linear system (Figure 1-5).

$$\mathbf{y}(\omega) = \mathbf{H}(\omega) \cdot \mathbf{u}(\omega) \quad (1.10)$$

where $\mathbf{H}(\omega)$ is called a frequency response function (FRF).

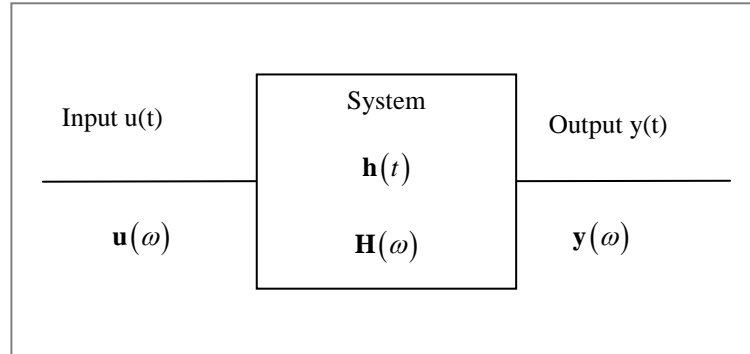


Figure 1-5 : Block diagram of a linear system

1.6 Non-proportional and non-classical damping and their measures

Modern aircraft structures are increasingly made of composite materials and/or bonded together with adhesives instead of being riveted or welded to achieve lighter weight. The complex aircraft designs need additional testing requirements due to increased use of composite materials and active control systems [4, 5]. Due to the development of the airframe structure materials the level of damping is increased related to modal aircraft structures of earlier years.

Most mechanical structures exhibit non-proportional and/or non-classical damping. Many methods have been used to extract real normal mode shapes of the structures using ground vibration testing.

General practice is to find Φ_R and Φ_L such that $\Phi_L^T \mathbf{M} \Phi_R$ and $\Phi_L^T \mathbf{K} \Phi_R$ are diagonal. Then $\Phi_L^T \mathbf{D} \Phi_R$ is assumed to be diagonal and its off diagonal terms are set to zero. The errors introduced by this assumption depend on how big the damping matrix values are.

1.6.1 Real value transformations SPEs

Structure preserving equivalences SPEs are more general than the conventional equivalences of equations (1.5), (1.6), (1.7) and they allow for most systems to be diagonalised. Diagonalising SPEs are used to decouple the original equations of motion such that the new system matrices themselves are diagonal. Matrices $\{\underline{\mathbf{T}}_R, \underline{\mathbf{T}}_L\}$ describe *diagonalising* SPEs for the system $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$. Each SPE is characterised by one $(2n \times 2n)$ matrix, $\underline{\mathbf{T}}_R$, and one $(2n \times 2n)$ matrix, $\underline{\mathbf{T}}_L$. If the original system matrices are real, then the transformation matrices and the diagonalised system matrices are also real.

1.7 Extracting normal modes from generally damped structures

The procedure of extracting the undamped natural frequencies and normal mode shapes of structures is known as normal mode force appropriation [13]. The main purpose of force appropriation techniques is to provide an estimate for the force distribution. The appropriated forces in modal vibration testing of complex structures

are used to excite a single normal mode of vibration at a time using multiple exciters [13-16].

A single force may be sufficient to excite a structure in order to measure pure normal mode shapes. However multi-exciter locations may be required where modes are close in frequency [14]. The number of exciters must be greater than or equal to the number of effective modes in the frequency range of interest [15]. Over many years in the aerospace industry, the normal mode force appropriation techniques have been used for ground vibration testing where accurate normal mode estimates are required for direct comparisons between appropriated normal mode responses and the results of finite element analysis [13, 15-17].

1.8 The Proposed Work

1.8.1 Motivation of the proposed work

Ground vibration testing (GVT) and Finite element method (FEM) are two techniques which are commonly performed to identify dynamic characteristics of structures. These characteristics include the resonant frequencies at which a structure naturally vibrates, the mode shapes associated with resonant vibration and the levels of damping governing the amplitude of resonance. The comparison of measured results with predicted results is important. Experimental measured data from GVT are not expected to match exactly the computed data from FEM. These discrepancies might be assessed, and many methods in the literature exist to evaluate the discrepancies between GVT and FEM modal parameters.

Any change occurring in the design parameters or modifications in the original structure causes associated changes of the dynamic characteristics of the structure. Cross orthogonality measures are also affected by the changes that occur in the system matrices. In order to correct and adjust the dynamic characteristics, eigenvalue and eigenvector derivatives are required. Also adjusting the cross orthogonality measures requires cross orthogonality measure derivatives. Both eigenvalue, eigenvector derivatives and cross orthogonality measures are used here to update the model system matrices.

1.8.2 The focus of the work

This study is mostly about the analysis of second order linear vibrating systems, and focuses on changes of the dynamic characteristics due to changes of the original system matrices. The main aims of this work are to develop model updating methods based on cross orthogonality measures for second order generally damped systems. The methods will be developed in state space in terms of real matrices involving diagonalising structure preserving equivalences SPEs. The work proposed in this thesis makes the following contribution areas:

1. Identification of dynamic characteristics of a GARTEUR-like aircraft structure. Modal analysis technique will be implemented to extract the modal parameters from a single column frequency response function (FRF). Finite element analysis will also be applied to determine the predicted model data of the GARTEUR-like aircraft structure.

2. Development of Craig Bampton transformations for undamped systems. These methods will be generalised in terms of SPEs to generally damped systems. The finite element analysis methods require the number of degrees of freedom to be reduced and the Craig Bampton method is implemented to decouple the substructure matrices.
3. Development of conventional eigenvalue and eigenvector derivative methods are implemented initially for undamped systems. The advanced methods will be generalised from the perspective of diagonalising SPEs to generally damped systems. The methods will be used for model updating.
4. Development of modal correlation methods based on cross orthogonality measures for undamped system. These ideas will be generalised to generally damped systems in terms of real matrix SPEs. The methods will be developed for ill-conditioned systems and a measure of the residuals will be developed. These approaches will be implemented for model updating methods.
5. Development of conventional model updating methods based on eigenvalues and eigenvectors for undamped systems. The ideas will be generalised through the utilisation of SPEs to generally damped systems. Development of model updating methods based on cross orthogonality and mutual orthogonality measures for undamped systems. These approaches will be generalised in terms of SPEs to generally damped systems.

1.9 Outline of Thesis

The remaining chapters of the thesis are summarised as follows:

Chapter 2 organises the currently available literature with regards to the various achievements of this project.

Chapter 3 presents a general overview of finite element modelling and focuses on 3-D aircraft structures.

Chapter 4 introduces some general information about modal testing and presents technique to determine the dynamic characteristics of aircraft structures.

Chapter 5 investigates new techniques to resolve undefined eigenvalue and eigenvector derivatives.

Chapter 6 discusses calculations of diagonalising SPEs for ill-conditioned systems using the Möbius transformation.

Chapter 7 focuses on the development of general modal correlation methods, investigates new methods for ill-conditioned systems and introduces a new measure for residuals.

Chapter 8 develops model updating methods and compares the developed modal correlation methods with alternative methods for model updating.

Chapter 9 generalises Craig-Bampton transformations for general damped second order systems.

Chapter 10 presents the conclusions of the methods presented and proposed future work.

CHAPTER 2. Literature Reviews

Comparison of measured results with the predicted results is important in design [18], validation of the structural dynamic models [19, 20], verification of the analytical model [21], and certification of airworthiness. Experimental measurements from physical tests are not expected to match exactly the analytical information provided from the finite element analysis. These discrepancies between measured and predicted modal information need to be assessed.

Modal correlation techniques are commonly used for evaluating the discrepancies between modal vectors measured using ground vibration testing (GVT) and modal vectors calculated from a finite element method (FEM). Most of the presented methods do not take into account the eigenvalues (frequencies) associated with eigenvectors (modes). The procedure of correcting the finite element model structure by processing vibration test data is called model updating. Many methods have been developed to correct the analytical models. The main purpose of model updating is to modify the model parameters in order to match the analytical data with measured data [22].

The aspiration of this project is to develop and generalise model updating methods for second order generally damped systems. These methods apply to generally damped systems. This chapter organises the currently available literature with regards to the various achievements of this project.

Section 2.1 demonstrates the structure of models of vibrating systems. Section 2.2 reviews the literature relevant to ground vibration testing of aircraft. Section 2.3 presents a definition of eigenvalues and eigenvectors for vibrating systems. Section 2.4 discusses frequency response function in frequency domain and in time domain. Section 2.5 focuses on coordinate transformations. Section 2.6 introduces the concept of structure preserving equivalences and diagonalising transformations for general damped systems. Section 2.7 concentrates on model reduction and modal expansion methods which are applied for large degrees of freedom systems. Section 2.8 explains the force appropriation methods that are required to excite a single pure normal mode of vibration. Section 2.9 considers modal correlation methods. Section 2.10 discusses model updating methods. It is appropriate to talk in brief about the basic vibration properties of general linear structures before commencing the various detailed aspects of this chapter.

2.1 Basic vibration properties and model structures

Figure 2-1 illustrates the structure of a dynamic model for a vibrating system and shows the vibration analysis.

2.1.1 Classes of vibrating systems

A physical system demonstrates two classes of vibration: Free and Forced [8]. Free vibration takes place when a system oscillates under the action of forces inherent in the system itself, where the external forces are set to zero in the equation of motion. The system will vibrate at one or more of its natural frequencies with associated

mode shapes. Forced vibration takes place when a system oscillates under the action of external forces. The system will vibrate at the excitation frequency, if the frequency of excitation coincides with one of the natural frequencies; the response is limited only by the degree of damping.

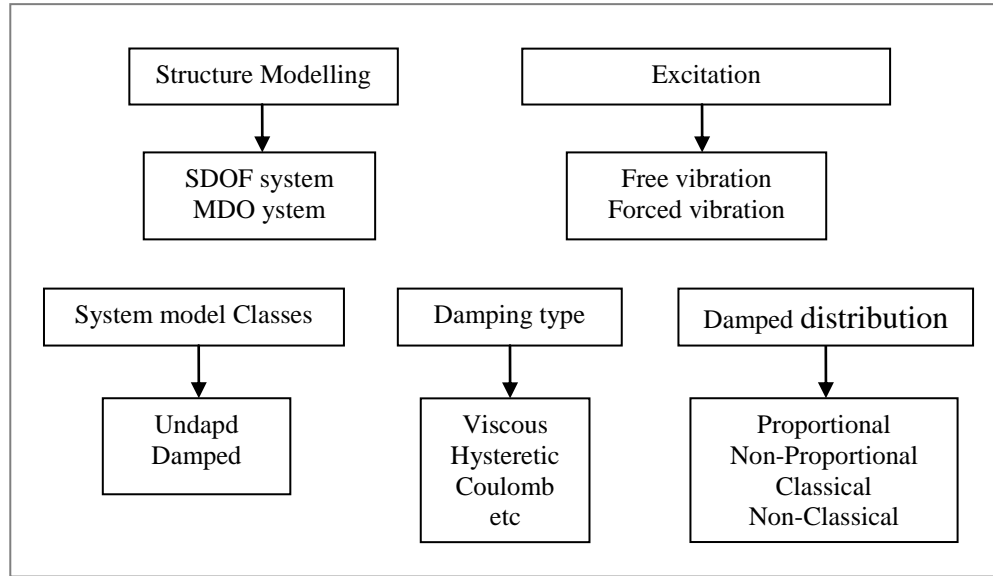


Figure 2-1 : Structures of dynamic model

2.1.2 Structure modelling

The degrees of freedom of the system are independent coordinates which are required to describe the motion of the system. The system could be modelled by a single-degree of freedom (SDOF) or multi-degrees of freedom (MDOF) [10]. The inputs represent the forces and the outputs represent the responses. The multi-degrees of freedom systems can be described by either single input and multi-outputs (SIMO) or multi-inputs and multi-outputs (MIMO) or single-input and single-output (SISO).

2.1.3 System model classes: type and distribution

The vibrating system can be described with two classes of system model; undamped or damped [10]. When the damping is very low, the system can be represented by mass and stiffness matrices only. Such systems are called an undamped-system. Real life systems are not undamped, since energy is always dissipated through friction and other resistances [8]. These resistances appear in various forms which are called Viscous, Hysteretic, Coulomb, and aerodynamic. The distribution of damping in the systems can be classified as proportional or non-proportional damping and classical or non-classical damping.

2.1.4 Categories of vibration analysis

Vibration analysis is divided into two categories: Theoretical (*predicted*) vibration analysis and Experimental (*measured*) vibration analysis [10]. Figure 2-2 shows the three stages of theoretical analysis. The structure model describes the structure physical characteristics in terms of mass, stiffness and damping properties. Performing theoretical modal analysis of the structural model result in the structure's behaviour as a set of natural frequencies (*eigenvalues*) associated with corresponding modal damping factors and vibration mode shapes (*eigenvectors*). The structure's response describes how the structure will respond under certain excitation conditions. This response model comprises a set of frequency response functions (FRFs).

It is possible to progress vibration analysis in the opposite direction [10]. Figure 2-3 defines the experimental vibration analysis which has three stages. The modal

properties and structural properties can be extracted from measured response properties such as measured frequency response functions.

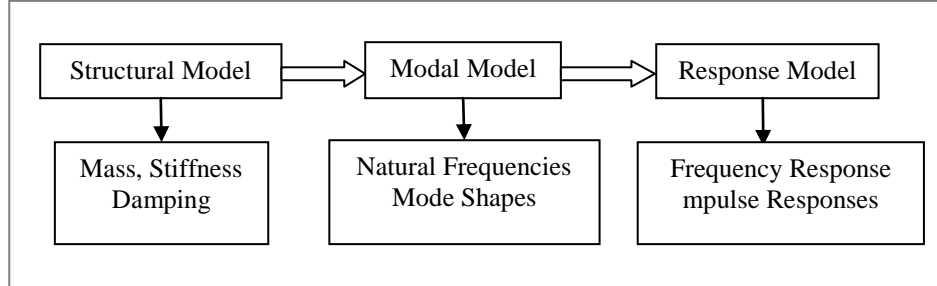


Figure 2-2 : Theoretical vibration analysis

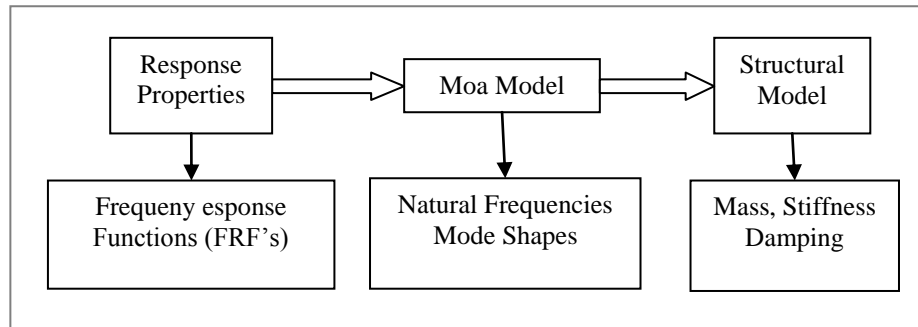


Figure 2-3 : Experimental vibration analysis

2.1.5 *The structure of models of vibrating systems*

Most dynamic models may be considered to comprise 3 stages [23, 24] as in Figure 2-4. The associated linearised equation of motion governing the vibration of any structure [23] appears in the form

$$\begin{aligned}
 \mathbf{f} &= \mathbf{S}_L \mathbf{u} \\
 \mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{D}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) &= \mathbf{f}(t) \\
 \mathbf{y} &= \mathbf{S}_R^T \mathbf{q}
 \end{aligned} \tag{2.1}$$

in which $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ are the system stiffness, damping and mass matrices-all $(n \times n)$, \mathbf{q} is a vector of displacement coordinates, \mathbf{f} is a vector of forces. Both vectors are

functions of time and the system has n degree of freedom. The vector \mathbf{y} represents the output displacements which has entries n_o , and \mathbf{u} represents vector of the corresponding input forces which has entries n_i . \mathbf{S}_R ($n \times n_o$) it controls how the generalised displacements are manifest in the output vector \mathbf{y} . \mathbf{S}_L ($n \times n_i$) it distributes the system input onto the full length force vector \mathbf{f} .

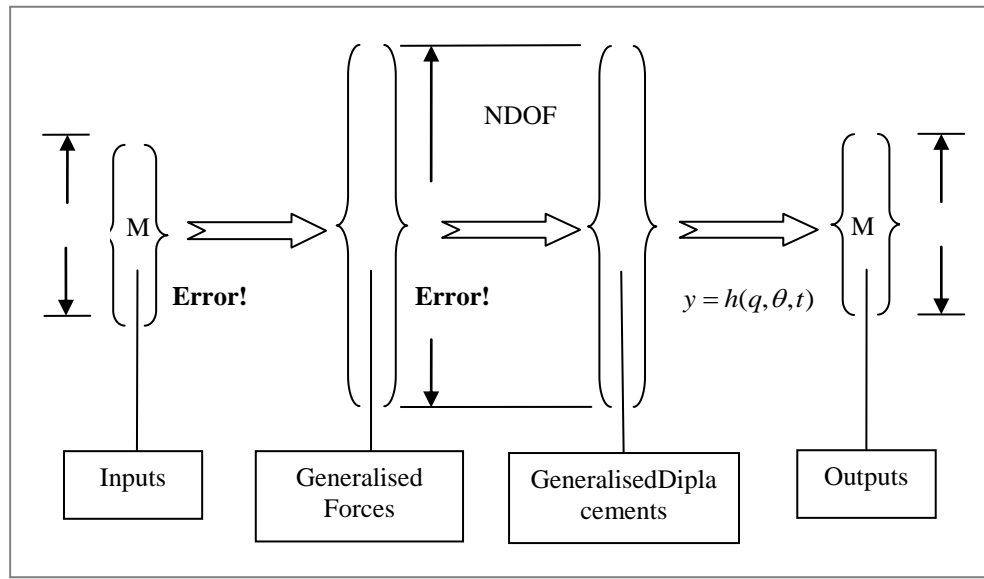


Figure 2-4 : Model structure in 3 stages

2.2 Ground Vibration Testing (GVT) of Aircraft

Experimental modal analysis results have played dominant roles in aerospace industries and engineering applications. Ground vibration testing is a process performed to determine the dynamic characteristics of mechanical structures [1, 25]. The dynamic characteristics can be extracted from GVT by causing the structure to shake and then measuring the responses. These characteristics comprise resonant

frequencies, mode shapes associated with resonant vibration and levels of damping governing the amplitude of resonance.

The comparison of measured data from GVT with predicted data from finite element method is important in many engineering applications. Invariably, these measurements do not match exactly what was predicted in a numerical model and indeed it is the mismatches that provide the information of interest. However, the mismatches have to be interpreted somehow and the way to do this interpretation is to understand how changing certain parameters in the numerical model will change the resonant frequencies and the mode-shapes.

The present study of ground vibration testing is timely since the new method deals very nicely with structures which have relatively high levels of damping and previous methods do not work at all well in these situations. The reason for increased levels of damping is that airframe structures are increasingly made of composite materials and/or bonded together with adhesives instead of being riveted or welded as used to be the case.

2.2.1 Validation of structural dynamic models

A GVT is a modal test conducted for the purpose of validating and improving a structural dynamic model of the airplane [5, 19, 20]. The structural dynamic model is validated by comparing its predicted modes with the experimental modes identified from the test [1]. This validated model is then modified to represent various in-flight configurations and a variety of failure conditions [19].

2.2.2 Verification of analytical models

Verification of analytical models using ground vibration test results is common practice in many engineering applications [21]. In [26], the ground vibration test results of a modified JetStar airplane are compared with the test results of the same airplane using multiple-input random excitation. The results indicated that the structure was sufficiently excited using the impulsive sine waveform.

2.2.3 Updating analytical models and aircraft certification

Ground vibration test also plays an important role in updating analytical models for new aircraft certification [6]. The cause of discrepancies between measured frequency responses and identified modal parameters are evaluated and identified by using GVT tests. Many methods exist for updating analytical models using experimental data. There is non-unique method for model updating due to different computational methods [20].

2.2.4 Identification of structural damage

Measured vibration data are used to identify structural damage [27]. Visual inspection may not detect and locate the damage. Structural damage such as cracks will cause the stiffness distribution in the structure to change [28]. Structural damage is identified using system dynamic properties [29]. Estimation of stiffness by using model updating defines the most likely damage at the largest reduction in stiffness [28].

2.2.5 *Ground vibration testing methods*

Different excitation techniques have been investigated for performing aircraft ground vibration testing [26, 30]. Many excitation waveforms have been used for both ground vibration and flight flutter testing throughout the years. These waveforms have included sine dwell, sine sweep, impact, and single and multiple-input random excitation [11, 31, 32].

In flight excitation has consisted mainly of impact, from exciter or control surface pulses, and sine sweeps using either wingtip, oscillating or aerodynamic vanes, existing control surfaces, oscillating masses, or rotating eccentric weights [33]. Ideally, a waveform that excites all structural modes simultaneously and in a short duration is required for in-flight applications because of the high cost of flight testing. The capability of this waveform to excite all the structural modes of interest on an aircraft structure was investigated by conducting a ground vibration test on a modified JetStar airplane [26].

2.3 Definitions of eigenvalues and eigenvectors

2.3.1 *Eigenvalues and eigenvectors for undamped systems*

The general governing equation of motion for an undamped second order system can be written as

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f} \quad (2.2)$$

The eigenvalues $\Lambda = \text{diag}(\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2)$ might be complex or real or mixed (real and complex). The corresponding right and left modal matrices $\{\Phi_R, \Phi_L\}$ diagonalise the system matrices $\{K, M\}$.

$$\Phi_L^T M \Phi_R = \Theta \quad (2.3)$$

$$\Phi_L^T K \Phi_R = \Lambda \Omega = \Psi \quad (2.4)$$

where $\{\Theta, \Psi, \Lambda\}$ are diagonal matrices, $\Theta = \text{diag}(\theta_1, \theta_2, \dots, \theta_n)$, $\Psi = \text{diag}(\psi_1, \psi_2, \dots, \psi_n)$.

The i^{th} eigenvalue λ_i with the corresponding right and left eigenvectors $\{\phi_{Ri}, \phi_{Li}\}$ satisfy the following equations

$$(\mathbf{K} - \lambda_i^2 \mathbf{M}) \phi_{Ri} = 0 \quad , \quad \phi_{Li}^T (\mathbf{K} - \lambda_i^2 \mathbf{M}) = 0 \quad (2.5)$$

Equations (2.3) and (2.4) can be written as

$$\phi_{Li}^T M \phi_{Ri} = \theta_i \quad (2.6)$$

$$\phi_{Li}^T K \phi_{Ri} = \lambda_i \theta_i = \psi_i \quad (2.7)$$

For symmetric undamped systems, the right and left modal matrices ($\Phi_R = \Phi_L$) are identical. The systems can usually be diagonalised by coordinate transformations equations (2.3) and (2.4) thus

$$\Phi_L^T M \Phi_R = \mathbf{I} \quad (2.8)$$

$$\Phi_L^T K \Phi_R = \Lambda \quad (2.9)$$

Matrices $\{\mathbf{I}, \Lambda\}$ are diagonal. Generalised mass and stiffness matrices respectively decouple multi-degrees of freedom systems into single-degree of freedom systems. Equation (2.2) can be written as

$$\ddot{\tilde{\mathbf{q}}} + \Lambda \tilde{\mathbf{q}} = \tilde{\mathbf{f}} \quad (2.10)$$

in which $\tilde{\mathbf{q}}$ is the displacement vector in the modal coordinate system.

$$\mathbf{q} = \Phi_R \tilde{\mathbf{q}} \quad , \quad \tilde{\mathbf{f}} = \Phi_L^T \mathbf{f} \quad (2.11)$$

2.3.2 Eigenvalues and eigenvectors for damped systems

The second order linear equation of motion for damped systems is expressed in equation (2.1). The right and left second order eigenvectors $\{\phi_{Ri}, \phi_{Li}\}$ (called short eigenvectors) satisfy the following equations

$$(\mathbf{K} + \lambda_i \mathbf{D} + \lambda_i^2 \mathbf{M}) \phi_{Ri} = 0 \quad (2.12)$$

$$\phi_{Li}^T (\mathbf{K} + \lambda_i \mathbf{D} + \lambda_i^2 \mathbf{M}) = 0 \quad (2.13)$$

2.3.2.1 Proportionally damped systems

For proportionally-damped systems, the mass normalised right and left second order modal matrices $\{\Phi_R, \Phi_L\}$ (called short modal matrices) are usually used to diagonalise mass, stiffness and damping matrices $\{\mathbf{M}, \mathbf{D}, \mathbf{K}\}$ as the following equations indicate

$$\Phi_L^T \mathbf{M} \Phi_R = \mathbf{I} \quad (2.14)$$

$$\Phi_L^T \mathbf{D} \Phi_R = \mathbf{\Gamma} \quad (2.15)$$

$$\Phi_L^T \mathbf{K} \Phi_R = \mathbf{\Lambda} \quad (2.16)$$

The proportional damping matrix \mathbf{D} is a linear combination of the mass and stiffness matrices $\{\mathbf{M}, \mathbf{K}\}$ this relationship is defined by

$$\mathbf{D} = (\alpha \mathbf{M} + \beta \mathbf{K}) \quad (2.17)$$

The generalised mass, damping and stiffness matrices $\{\mathbf{I}, \mathbf{\Gamma}, \mathbf{\Lambda}\}$ respectively in equation (2.14), (2.15) and (2.16) are diagonal matrices which decouple multi-degrees of freedom systems with n DoFs into n single-degree of freedom systems. Substituting equation (2.11) into equation (2.1) yields

$$\begin{aligned}\tilde{\mathbf{f}} &= \mathbf{\Phi}_L^T \mathbf{f} \\ \ddot{\tilde{\mathbf{q}}} + \mathbf{\Gamma} \dot{\tilde{\mathbf{q}}} + \mathbf{\Lambda} \tilde{\mathbf{q}} &= \tilde{\mathbf{f}} \\ \mathbf{q} &= \mathbf{\Phi}_R \tilde{\mathbf{q}}\end{aligned}\tag{2.18}$$

The definition of proportional damping is extremely constrictive. There are many cases where one may diagonalise the system matrices and the definition delineated in equation (2.17) does not hold. This exists when the system is classically-damped.

2.3.2.2 Classically damped systems

Caughey and O’Kelly [9] defined classically damped systems with symmetric coefficient matrices which possess classical normal modes. The system matrices $\{\mathbf{M}, \mathbf{D}, \mathbf{K}\}$ may be diagonalised if and only if one of the conditions in equation (1.8) are satisfied

The required criterions in equation (1.8) are sufficient but not necessary. Based on this result, the series of representative damping is a necessary and sufficient condition for the existence of classical normal modes and this series is known as the “Caughey series”.

$$\mathbf{D} = \mathbf{M} \sum_{j=0}^{n-1} \alpha_j [\mathbf{M}^{-1} \mathbf{K}]^j\tag{2.19}$$

Ma and Caughey [34] proved that, in the case of non-symmetric system matrices; equation (2.19) still describes the necessary and sufficient condition for simultaneous diagonalisation of the system matrices by an equivalent transformation.

$$\Phi_L^T \mathbf{M} \Phi_R = \mathbf{M}_D \quad (2.20)$$

$$\Phi_L^T \mathbf{D} \Phi_R = \mathbf{D}_D \quad (2.21)$$

$$\Phi_L^T \mathbf{K} \Phi_R = \mathbf{K}_D \quad (2.22)$$

2.3.2.3 *Non-proportional or non classically damped systems*

Most real structures having non-zero damping are expected to be non-proportionally-damped or non-classically-damped systems and these have *complex modes* as a result. The usual view taken of the modes of second-order systems begins with the supposition that \mathbf{M} is invertible. Then, equation (2.1) can be written in a state-space form as

$$\begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} - \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} \quad (2.23)$$

When no forcing is present, this becomes

$$\underline{\mathbf{D}}\mathbf{x} - \underline{\mathbf{K}}\dot{\mathbf{x}} = \mathbf{0} \quad (2.24)$$

where $\underline{\mathbf{D}}$ and $\underline{\mathbf{K}}$ are $(2n \times 2n)$ matrices, \mathbf{x} is the so-called *state-vector* and the definitions are evident from comparison of equation (2.23) with equation (2.24). A generalised eigenvalue-eigenvector problem defined using matrices $\underline{\mathbf{D}}$ and $\underline{\mathbf{K}}$ above yields the triple of $(2n \times 2n)$ matrices $\{\Phi_R, \Phi_L, \Lambda\}$ which represent the right and left modal matrices and eigenvalues.

The damping matrix \mathbf{D} is not a linear combination of the mass and stiffness matrix $\{\mathbf{M}, \mathbf{K}\}$ [35]. Real eigenvectors cannot decouple the equations of motion in which the generalised damping matrix $(\Phi_L^T \mathbf{D} \Phi_R)$ is non diagonal.

The eigenvalues with their associated eigenvectors can be all real. The right and left modal matrices can be written in the following forms respectively.

$$\underline{\Phi}_R = \begin{bmatrix} \Phi_R \\ \Phi_R \Lambda \end{bmatrix} \quad (2.25)$$

$$\underline{\Phi}_L = \begin{bmatrix} \Phi_L \\ \Phi_L \Lambda \end{bmatrix} \quad (2.26)$$

Λ is (usually) a diagonal $(2n \times 2n)$ matrix and $\{\Phi_R, \Phi_L\}$ are $(n \times 2n)$ matrices. obeying

$$\mathbf{M} \Phi_R \Lambda^2 + \mathbf{D} \Phi_R \Lambda + \mathbf{K} \Phi_R = \mathbf{0} \quad (2.27)$$

$$\mathbf{M}^T \Phi_L \Lambda^2 + \mathbf{D}^T \Phi_L \Lambda + \mathbf{K}^T \Phi_L = \mathbf{0} \quad (2.28)$$

The pairs of matrices (Φ_R, Λ) and (Φ_L, Λ) are called standard pairs for equation (2.23) if $\underline{\Phi}_R$ and $\underline{\Phi}_L$ from equations (2.25)-(2.26) are non-singular and equations (2.27)-(2.28) are obeyed.

2.3.2.4 *Extract real eigenvectors from real and complex conjugate eigenvectors*

The eigenvalues with their associated eigenvectors can be either all real as in equations (2.25)-(2.26) or all complex conjugate or a mixture of real and complex conjugate [36] that can be written in the following forms respectively.

$$\underline{\Phi_R} = \left[\begin{array}{c|c} \Phi_{Rc} & \Phi_{Rc}^* \\ \hline \Phi_{Rc}\Lambda_c & \Phi_{Rc}^*\Lambda_c^* \end{array} \right] \quad (2.29)$$

$$\underline{\Phi_L} = \left[\begin{array}{c|c} \Phi_{Lc} & \Phi_{Lc}^* \\ \hline \Phi_{Lc}\Lambda_c & \Phi_{Lc}^*\Lambda_c^* \end{array} \right] \quad (2.30)$$

$$\underline{\Phi_R} = \left[\begin{array}{cc|cc} \Phi_{Rc} & \Phi_{Rr1} & \Phi_{Rc}^* & \Phi_{Rr2} \\ \hline \Phi_{Rc}\Lambda_c & \Phi_{Rr1}\Lambda_{r1} & \Phi_{Rc}^*\Lambda_c^* & \Phi_{Rr2}\Lambda_{r2} \end{array} \right] \quad (2.31)$$

$$\underline{\Phi_L} = \left[\begin{array}{cc|cc} \Phi_{Lc} & \Phi_{Lr1} & \Phi_{Lc}^* & \Phi_{Lr2} \\ \hline \Phi_{Lc}\Lambda_c & \Phi_{Lr1}\Lambda_{r1} & \Phi_{Lc}^*\Lambda_c^* & \Phi_{Lr2}\Lambda_{r2} \end{array} \right] \quad (2.32)$$

The superscript notation “*” indicates a complex conjugate. The subscript “c” in equations (2.29)-(2.32) indicates complex. $\{\Phi_{Rc}, \Phi_{Lc}\}$ are complex modal matrices. The size of real modal matrices $\{\Phi_{Rr1}, \Phi_{Rr2}, \Phi_{Lr1}, \Phi_{Lr2}\}$ in equations (2.31) and (2.32) is $(n - c_c)$ where “ c_c ” is the size of the complex modal matrices $\{\Phi_{Rc}, \Phi_{Lc}\}$.

In general, $\{\Phi_L, \Lambda, \Phi_R\}$ may be complex as in equations (2.29) and (2.30) but because the system matrices themselves are invariably real-valued for all practical cases, every complex entry in Λ is joined by its complex conjugate and the same applies to every complex column in $\underline{\Phi_L}$ and in $\underline{\Phi_R}$. In total, it is clear that $\{\Phi_L, \Lambda, \Phi_R\}$ contain $2n(2n+1)$ potentially-distinct real numbers whereas the original system matrices, $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$, contain only $3n^2$ real numbers at most. Certain constraints apply to the modal information for such systems and these constraints are well-understood when all system eigenvalues are complex [36-38].

$$\underline{\Phi_{Rreal}} = \left[\begin{array}{c|c} \Phi_R & \Phi_R^* \\ \hline \Phi_R\Lambda & \Phi_R^*\Lambda^* \end{array} \right] \left[\begin{array}{c|c} \mathbf{I} & \mathbf{J} \\ \hline \mathbf{I} & -\mathbf{J} \end{array} \right] = \left[\begin{array}{c|c} (\Phi_R + \Phi_R^*) & (\Phi_R - \Phi_R^*) \\ \hline (\Phi_R\Lambda + \Phi_R^*\Lambda^*) & (\Phi_R\Lambda - \Phi_R^*\Lambda^*) \end{array} \right] \quad (2.33)$$

$$\underline{\Phi}_{Lreal} = \left[\begin{array}{c|c} \Phi_L & \Phi_L^* \\ \hline \Phi_L \Lambda & \Phi_L^* \Lambda^* \end{array} \right] \left[\begin{array}{c|c} \mathbf{I} & \mathbf{J} \\ \hline \mathbf{I} & -\mathbf{J} \end{array} \right] = \left[\begin{array}{c|c} (\Phi_L + \Phi_L^*) & (\Phi_L - \Phi_L^*) \\ \hline (\Phi_L \Lambda + \Phi_L^* \Lambda^*) & (\Phi_L \Lambda - \Phi_L^* \Lambda^*) \end{array} \right] \quad (2.34)$$

where \mathbf{I} is $(n \times n)$ identity matrix and \mathbf{J} is $(\mathbf{I} \times j)$ and $j = \sqrt{-1}$.

Adhiakri [39] derived conditions for the existence of classical normal modes in nonviscously damped asymmetric linear multiple-degree of freedom systems. If the system matrices are positive definite, damping is a time domain function, and there exist two non-singular matrices $\Phi_L \in \mathbb{R}^{n \times n}$ and $\Phi_R \in \mathbb{R}^{n \times n}$ such that $\Phi_L^T \mathbf{M} \Phi_R$, $\Phi_L^T \mathbf{K} \Phi_R$ and $\Phi_L^T \mathbf{D}(t) \Phi_R \forall t$ are all real diagonal matrices then the following are equivalent

$$\begin{aligned} \mathbf{D}(t) \mathbf{M}^{-1} \mathbf{K} &= \mathbf{K} \mathbf{M}^{-1} \mathbf{D}(t) \\ \mathbf{D}(t) \mathbf{K}^{-1} \mathbf{M} &= \mathbf{M} \mathbf{K}^{-1} \mathbf{D}(t) \\ \mathbf{M} \mathbf{D}(t)^{-1} \mathbf{K} &= \mathbf{K} \mathbf{D}(t)^{-1} \mathbf{M} \end{aligned} \quad (2.35)$$

The $\{\underline{\Phi}_R, \underline{\Phi}_L\}$ modal matrices may be a mixture of real and complex conjugate. The real modal matrices can be achieved by rearranging the matrices in equations (2.31), (2.32) and multiplying by a \mathbf{I}_J matrix. Equations (2.31), (2.32) can be written as

$$\begin{aligned} \underline{\Phi}_{Rreal} &= \left[\begin{array}{c|c} \Phi_{Rc} & \Phi_{Rr1} \\ \hline \Phi_{Rc} \Lambda_c & \Phi_{Rr1} \Lambda_{r1} \end{array} \right] \left[\begin{array}{c|c} \Phi_{Rc}^* & \Phi_{Rr2} \\ \hline \Phi_{Rc}^* \Lambda_c^* & \Phi_{Rr2} \Lambda_{r2} \end{array} \right] \left[\begin{array}{cc|cc} \mathbf{I}_c & 0 & \mathbf{J}_c & 0 \\ 0 & \mathbf{I}_{n-c} & 0 & 0 \\ \hline \mathbf{I}_c & 0 & -\mathbf{J}_c & 0 \\ 0 & 0 & 0 & \mathbf{I}_{n-c} \end{array} \right] \\ &= \left[\begin{array}{c|c} (\Phi_{Rc} + \Phi_{Rc}^*) & \Phi_{Rr1} \\ \hline (\Phi_{Rc} \Lambda_c + \Phi_{Rc}^* \Lambda_c^*) & \Phi_{Rr1} \Lambda_{r1} \end{array} \right] \left[\begin{array}{c|c} (\Phi_{Rc}^* - \Phi_{Rc}) & \Phi_{Rr2} \\ \hline (\Phi_{Rc}^* \Lambda_c^* - \Phi_{Rc} \Lambda_c) & \Phi_{Rr2} \Lambda_{r2} \end{array} \right] \end{aligned} \quad (2.36)$$

$$\begin{aligned} \underline{\Phi}_{Lreal} &= \begin{bmatrix} \Phi_{Lc} & \Phi_{Lr1} & \Phi_{Lc}^* & \Phi_{Lr2} \\ \Phi_{Lc}\Lambda_c & \Phi_{Lr1}\Lambda_{r1} & \Phi_{Lc}^*\Lambda_c^* & \Phi_{Lr2}\Lambda_{r2} \end{bmatrix} \begin{bmatrix} \mathbf{I}_c & 0 & \mathbf{J}_c & 0 \\ 0 & \mathbf{I}_{n-c} & 0 & 0 \\ \hline \mathbf{I}_c & 0 & -\mathbf{J}_c & 0 \\ 0 & 0 & 0 & \mathbf{I}_{n-c} \end{bmatrix} \\ &= \begin{bmatrix} (\Phi_{Lc} + \Phi_{Lc}^*) & \Phi_{Lr1} & (\Phi_{Lc}^* - \Phi_{Lc}) & \Phi_{Lr2} \\ \hline (\Phi_{Lc}\Lambda_c + \Phi_{Lc}^*\Lambda_c^*) & \Phi_{Lr1}\Lambda_{r1} & (\Phi_{Lc}^*\Lambda_c^* - \Phi_{Lc}\Lambda_c) & \Phi_{Lr2}\Lambda_{r2} \end{bmatrix} \end{aligned} \quad (2.37)$$

where $\mathbf{I}_j = \begin{bmatrix} \mathbf{I}_c & 0 & \mathbf{J}_c & 0 \\ 0 & \mathbf{I}_{n-c} & 0 & 0 \\ \hline \mathbf{I}_c & 0 & -\mathbf{J}_c & 0 \\ 0 & 0 & 0 & \mathbf{I}_{n-c} \end{bmatrix}$, \mathbf{I}_c is a $(c_c \times c_c)$ identity matrix and \mathbf{J}_c is an $(\mathbf{I}_c \times j)$

and $j = \sqrt{-1}$.

2.4 Frequency Response Function

2.4.1 Frequency domain calculation using linear models

Equation (2.1) can be written in the frequency domain as

$$\begin{aligned} \mathbf{f} &= \mathbf{S}_L \mathbf{u} \\ (\mathbf{K} + i\omega\mathbf{D} - \omega^2\mathbf{M})\mathbf{q} &= \mathbf{f} \\ \mathbf{y} &= \mathbf{S}_R^T \mathbf{q} \end{aligned} \quad (2.38)$$

Rewrite equation (2.38) as

$$\begin{aligned} \mathbf{q} &= (\mathbf{K} + i\omega\mathbf{D} - \omega^2\mathbf{M})^{-1} \mathbf{f} \\ \mathbf{y} &= \mathbf{S}_R^T (\mathbf{K} + i\omega\mathbf{D} - \omega^2\mathbf{M})^{-1} \mathbf{S}_L \mathbf{u} \end{aligned} \quad (2.39)$$

Equation (2.38) can be transformed using the modal transformation $\{\Phi_R, \Phi_L\}$.

Substituting equation (2.11) into equation (2.38) yields

$$\begin{aligned} \bar{\mathbf{f}} &= \Phi_L^T \mathbf{S}_L \mathbf{u} \\ (\omega_n^2 + i\omega\Gamma - \omega^2\mathbf{I})\bar{\mathbf{q}} &= \bar{\mathbf{f}} \\ \mathbf{y} &= \mathbf{S}_R^T \Phi_R \bar{\mathbf{q}} \end{aligned} \quad (2.40)$$

where $\Gamma = \{2\zeta\omega_n\}$. Rewrite equation (2.40) as

$$\begin{aligned}\bar{\mathbf{q}} &= \left(\omega_n^2 + i\omega\Gamma - \omega^2\mathbf{I} \right)^{-1} \bar{\mathbf{f}} \\ \mathbf{y} &= \mathbf{S}_R^T \Phi_R \left(\omega_n^2 + i\omega\Gamma - \omega^2\mathbf{I} \right)^{-1} \Phi_L^T \mathbf{S}_L \mathbf{u}\end{aligned}\quad (2.41)$$

The measured FRF ($\mathbf{H}(\omega)$) for the system in terms of displacement called admittance, can be expressed by two equations (2.39) and (2.41)

$$\begin{aligned}\mathbf{H}(\omega) &= \mathbf{S}_R^T \left(\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{D} \right)^{-1} \mathbf{S}_L \\ &= \mathbf{S}_R^T \Phi_R \left(\omega_n^2 - \omega^2\mathbf{I} + i\omega\Gamma \right)^{-1} \Phi_L^T \mathbf{S}_L \\ &= \tilde{\Phi}_R \left(\omega_n^2 - \omega^2\mathbf{I} + i\omega\Gamma \right)^{-1} \tilde{\Phi}_L^T\end{aligned}\quad (2.42)$$

Equation (2.42) represents the complex frequency response function which forms the real and imaginary parts of $\mathbf{H}(\omega)$. The real part of $\mathbf{H}(\omega)$ expresses the undamped eigenvalues problem

$$\mathbf{H}(\omega) = (\mathbf{A}(\omega) + i\mathbf{B}(\omega)) \quad (2.43)$$

where \mathbf{A} is the real part of $\mathbf{H}(\omega)$. \mathbf{B} is the imaginary part of $\mathbf{H}(\omega)$.

2.4.2 Time domain simulation using linear models

2.4.2.1 General methods for time domain simulation

There are several numerical integration methods for solving equation of motion (2.1) in a time domain simulation such as Euler's method, time-stepping, time-marching (MATLAB ode45), finite difference, and the Runge-Kutta method [40]. The equation of motion (2.1) for linear vibrating systems can be written in the time domain as

$$\ddot{\mathbf{q}}(t) = \mathbf{M}^{-1} \left(\mathbf{f}(t) - (\mathbf{D}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t)) \right) \quad (2.44)$$

We can rewrite equation of motion (2.1) in state space as

$$\begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} - \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix} \quad (2.45)$$

Let $\underline{\mathbf{x}}$ represent a vector of state for vibrating systems

$$\dot{\underline{\mathbf{x}}} = \begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix}, \quad \underline{\mathbf{x}} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}, \quad \underline{\mathbf{f}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix} \quad (2.46)$$

we can rearrange equations (2.45) and (2.46) as

$$\begin{aligned} \underline{\mathbf{f}} &= \begin{bmatrix} \mathbf{0} \\ \mathbf{S}_L \end{bmatrix} \underline{\mathbf{u}} \\ \dot{\underline{\mathbf{x}}} &= \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{S}_L \end{bmatrix} \underline{\mathbf{u}} + \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix} \underline{\mathbf{x}} \\ \underline{\mathbf{y}} &= \begin{bmatrix} \mathbf{S}_R^T & \mathbf{0} \end{bmatrix} \underline{\mathbf{x}} \end{aligned} \quad (2.47)$$

The rate vector $\dot{\underline{\mathbf{x}}}$ in time domain simulation requires that we write

$$\dot{\underline{\mathbf{x}}} = f(\underline{\mathbf{x}}(t), \underline{\mathbf{u}}(t), t) \quad (2.48)$$

If we know $\underline{\mathbf{u}}(t) \forall 0 < t < \tau$ and $\underline{\mathbf{x}}(0)$, then the numerical integration is possible.

2.4.2.2 Runge-Kutta method

The Runge-Kutta procedure [40] is popular since it is self starting and results in good accuracy. In the Runge-Kutta method the second order differential equation is first reduced into two first order equations.

Let $\mathbf{x} = \dot{\mathbf{q}}$, the equation (2.44) is reduced to the following two first order equations.

$$\begin{aligned} \dot{\mathbf{q}} &= \mathbf{x} \\ \dot{\mathbf{x}} &= f(\mathbf{q}, \mathbf{x}, t) \end{aligned} \quad (2.49)$$

Both \mathbf{q} and \mathbf{x} can be expressed in terms of the Taylor series and the time increment is $h = \Delta t$. The first derivative is replaced by an average slope and ignoring higher order derivatives [41] gives

$$q = q_i + \left(\frac{dq}{dt} \right)_{iav} h \quad , \quad x = x_i + \left(\frac{dx}{dt} \right)_{iav} h \quad (2.50)$$

The average in the interval h becomes

$$\begin{aligned} \left(\frac{dq}{dt} \right)_{iav} &= \frac{1}{6} \left(\left(\frac{dq}{dt} \right)_{t_i} + 2 \left(\frac{dq}{dt} \right)_{t_i+h/2} + 2 \left(\frac{dq}{dt} \right)_{t_i+h/2} + \left(\frac{dq}{dt} \right)_{t_i+h} \right) \\ \left(\frac{dx}{dt} \right)_{iav} &= \frac{1}{6} \left(\left(\frac{dx}{dt} \right)_{t_i} + 2 \left(\frac{dx}{dt} \right)_{t_i+h/2} + 2 \left(\frac{dx}{dt} \right)_{t_i+h/2} + \left(\frac{dx}{dt} \right)_{t_i+h} \right) \end{aligned} \quad (2.51)$$

where $\frac{dq}{dt} = X$, $\frac{dx}{dt} = F$.

The centre term of the above equation is split into two terms and four values of t, q, x , and f are computed for each point i as shown in the Table 2-1. The next point of calculation $t_{i+1} = (t_i + h)$ is

$$\begin{aligned} q_{i+1} &= q_i + \frac{h}{6} (X_1 + X_2 + X_3 + X_4) \\ x_{i+1} &= x_i + \frac{h}{6} (F_1 + F_2 + F_3 + F_4) \end{aligned} \quad (2.52)$$

t	q	x	$f = \ddot{x} = \ddot{q}$
$T_1 = t_i$	$Q_1 = q_i$	$X_1 = x_i$	$F_1 = f(T_1, Q_1, X_1)$
$T_2 = t_i + \frac{h}{2}$	$Q_2 = q_i + X_1 \frac{h}{2}$	$X_2 = x_i + F_1 \frac{h}{2}$	$F_2 = f(T_2, Q_2, X_2)$
$T_3 = t_i + \frac{h}{2}$	$Q_3 = q_i + X_2 \frac{h}{2}$	$X_3 = x_i + F_2 \frac{h}{2}$	$F_3 = f(T_3, Q_3, X_3)$
$T_4 = t_i + h$	$Q_4 = q_i + X_3 h$	$X_4 = x_i + F_3 h$	$F_4 = f(T_4, Q_4, X_4)$

Table 2-1 : Average of the interval h

2.4.2.3 Duhamel's integral

The second order differential equation of motion for linear systems can be written in the form equation (2.2). Duhamel's integral method [10] is based on the ability to

compute the response of a linear system to a time-varying forcing function. The arbitrary forcing function $\mathbf{f}(t)$ can be regarded as a series of impulsive forces $\mathbf{f}(\tau)$ acting over a short interval $d\tau$ see Figure 2-5.

The force produces a short duration impulse $\mathbf{f}(\tau)d\tau$ (Figure 2-5). The response of the system to this impulse for all $t > \tau$ is defined as the system's unit Impulse Response Function (IRF) and has a direct relationship to the Frequency Response Function (FRF). IRF is written as $\mathbf{H}(t-\tau)$ which is called Kernel function. The response for a system at time t can be written as [10]:

$$d\mathbf{q}(t) = \mathbf{H}(t-\tau)\mathbf{f}(\tau)d\tau \quad (2.53)$$

and the total response of the system can be found by integrating all the incremental responses which is called convolution integrals as follows

$$\mathbf{q}(t) = \int_{-\infty}^{\infty} \mathbf{H}(t-\tau)\mathbf{f}(\tau)d\tau \quad (2.54)$$

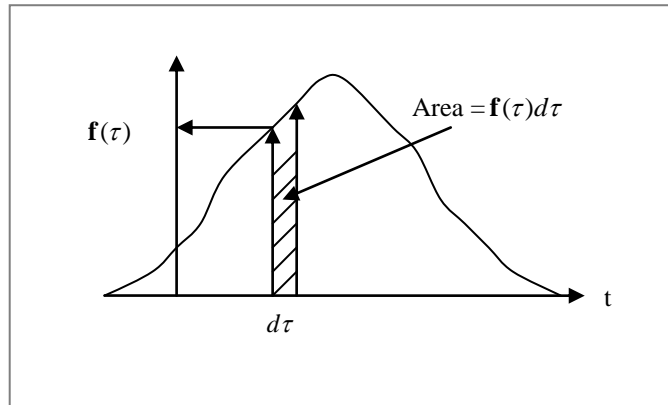


Figure 2-5 : Arbitrary forcing function

2.5 Coordinate transformation

The dynamic system is described by $(n \times n)$ matrices $\{\mathbf{M}_o, \mathbf{D}_o, \mathbf{K}_o\}$, which represent mass, damping and stiffness components of the system. Any pair of $(n \times n)$ matrices $\{\mathbf{T}_L, \mathbf{T}_R\}$ can be used to define a coordinate transformation [42] according to the following equations

$$\begin{aligned}\mathbf{f}_o &= \mathbf{S}_{Lo} \mathbf{u} \\ \mathbf{M}_o \ddot{\mathbf{q}}_o(t) + \mathbf{D}_o \dot{\mathbf{q}}_o(t) + \mathbf{K}_o \mathbf{q}_o(t) &= \mathbf{f}_o(t) \\ \mathbf{y} &= \mathbf{S}_{Ro}^T \mathbf{q}_o\end{aligned}\tag{2.55}$$

These transformations $\{\mathbf{T}_R, \mathbf{T}_L\}$ replace the original vectors $\{\mathbf{q}_o, \mathbf{f}_o\}$ with new vectors $\{\mathbf{q}_N, \mathbf{f}_N\}$.

$$\mathbf{q}_o = \mathbf{T}_R \mathbf{q}_N, \quad \mathbf{f}_N = \mathbf{T}_L^T \mathbf{f}_o, \quad \mathbf{S}_{RN}^T = (\mathbf{S}_{Ro}^T \mathbf{T}_R), \quad \mathbf{S}_{LN} = (\mathbf{T}_L^T \mathbf{S}_{Lo})\tag{2.56}$$

Then, the standard equation of motion can be written as

$$\begin{aligned}\mathbf{f}_N &= \mathbf{S}_{LN} \mathbf{u} \\ \mathbf{K}_N \mathbf{q}_N + \mathbf{D}_N \dot{\mathbf{q}}_N + \mathbf{M}_N \ddot{\mathbf{q}}_N &= \mathbf{f}_N \\ \mathbf{y} &= \mathbf{S}_{RN}^T \mathbf{q}_N\end{aligned}\tag{2.57}$$

Subscripts o and N indicate the *original* and *new* system. Subscripts L and R distinguish between left eigenvectors (or left transformations) and right eigenvectors (or right transformations). It is well known that the spectrum of some original system, $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$, is identical to the spectrum of some new system $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ if there are some invertible matrices, $\{\mathbf{T}_R, \mathbf{T}_L\}$ such that the dynamic models can be written as

$$\mathbf{T}_L^T \mathbf{K}_o \mathbf{T}_R = \mathbf{K}_N\tag{2.58}$$

$$\mathbf{T}_L^T \mathbf{D}_o \mathbf{T}_R = \mathbf{D}_N\tag{2.59}$$

$$\mathbf{T}_L^T \mathbf{M}_o \mathbf{T}_R = \mathbf{M}_N \quad (2.60)$$

The relationship between two systems is referred as a *conventional* equivalence. It is also well known that given any arbitrary system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$, it is not usually possible to find invertible matrices, $\{\mathbf{T}_R, \mathbf{T}_L\}$ such that $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ are all diagonal. Caughey and O’Kelly [9] expressed one sufficient criterion as in equation (1.8) to satisfy equations (2.58)-(2.60) in which $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ are diagonal.

If \mathbf{T}_R and \mathbf{T}_L have fewer columns than rows ($n > m$), then, the coordinate transformation reduces the number of system degrees of freedom. In this case the transformation is not invertible it is a model reducing transformation. The concept of model reduction is based on reducing the number of degrees of freedom of the system. The equation of motion can be written as

$$\begin{bmatrix} \mathbf{M}_{rr} & \mathbf{M}_{re} \\ \mathbf{M}_{er} & \mathbf{M}_{ee} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_r \\ \ddot{\mathbf{q}}_e \end{bmatrix} + \begin{bmatrix} \mathbf{D}_{rr} & \mathbf{D}_{re} \\ \mathbf{D}_{er} & \mathbf{D}_{ee} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_r \\ \dot{\mathbf{q}}_e \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{rr} & \mathbf{K}_{re} \\ \mathbf{K}_{er} & \mathbf{K}_{ee} \end{bmatrix} \begin{bmatrix} \mathbf{q}_r \\ \mathbf{q}_e \end{bmatrix} = \begin{bmatrix} \mathbf{f}_r \\ \mathbf{f}_e \end{bmatrix} \quad (2.61)$$

where subscripts ‘ r ’ and ‘ e ’ represent the reduced and eliminated degrees of freedom respectively. All model reduction strategies for undamped systems use a real-valued coordinate transformation. The transformation matrices \mathbf{T}_R and \mathbf{T}_L replace the original vectors $\{\mathbf{q}_o, \mathbf{f}_o\}$ with new vectors $\{\mathbf{q}_r, \mathbf{f}_r\}$. The original (full length) vector of coordinates is expressed linearly in terms of a much smaller vector of coordinates [43].

$$\mathbf{q}_o = \mathbf{T}_R \mathbf{q}_r \quad , \quad \mathbf{f}_r = \mathbf{T}_L^T \mathbf{f}_o \quad (2.62)$$

The equation of motion (2.55) for the reduced system can be written as

$$\begin{aligned}
\mathbf{f}_r &= \mathbf{S}_{Lr} \mathbf{u} \\
\mathbf{K}_r \mathbf{q}_r + \mathbf{D}_r \dot{\mathbf{q}}_r + \mathbf{M}_r \ddot{\mathbf{q}}_r &= \mathbf{f}_r \\
\mathbf{y} &= \mathbf{S}_{Rr}^T \mathbf{q}_r
\end{aligned} \tag{2.63}$$

in which $\mathbf{M}_r, \mathbf{K}_r, \mathbf{D}_r$ are the reduced system matrices

$$\mathbf{M}_r = \mathbf{T}_L^T \mathbf{M}_o \mathbf{T}_R, \quad \mathbf{K}_r = \mathbf{T}_L^T \mathbf{K}_o \mathbf{T}_R, \quad \mathbf{D}_r = \mathbf{T}_L^T \mathbf{D}_o \mathbf{T}_R \tag{2.64}$$

2.6 Structure Preserving Equivalences (SPEs)

A “structure-preserving equivalence” [44] in the sense intended here is a mapping between the stiffness, damping and mass matrices describing some initial second-order system and the corresponding three matrices of another second-order system having identical spectrum.

These transformations are more general than the conventional equivalences of equations (2.58)-(2.60) and they allow for most systems to be diagonalised. They were first exposed in [42, 44]-but they were referred to as *structure-preserving transformations* in these papers. Each SPE is characterised by one $(2N \times 2N)$ matrix, $\underline{\mathbf{T}}_L$, acting on the left hand side and one $(2N \times 2N)$ matrix, $\underline{\mathbf{T}}_R$, acting on the right hand side in much the same way that equations (2.58)-(2.60) involve the $(N \times N)$ matrices $\{\mathbf{T}_L, \mathbf{T}_R\}$. If the original system matrices are real, then the transformation matrices and the diagonalised system matrices are also real.

Garvey *et al* [42, 44] presented a more general approach to coordinate transformations for second-order systems (extended to higher order systems in [45,

46]). These more general coordinate transformations are referred to here as SPEs and they can be understood as left and right transformation matrices. They preserve the structure of *Lancaster augmented matrices* (LAMs) and the eigenvalues of these systems are identical. The structure of LAMs for any second order system are presented in the form

$$\underline{\mathbf{M}} = \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix}, \quad \underline{\mathbf{D}} = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix}, \quad \underline{\mathbf{K}} = \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix}$$

An alternative view of the modal information of any second-order system is possible through the concept of structure-preserving transformations (SPTs) [47]. Two different real-valued second-order systems, $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ and $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ are related by a SPT if there exist real-valued $(n \times n)$ matrices $\{\mathbf{W}_L, \mathbf{X}_L, \mathbf{Y}_L, \mathbf{Z}_L\}$, $\{\mathbf{W}_R, \mathbf{X}_R, \mathbf{Y}_R, \mathbf{Z}_R\}$ such that

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K}_o \\ \mathbf{K}_o & \mathbf{D}_o \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_N \\ \mathbf{K}_N & \mathbf{D}_N \end{bmatrix} \quad (2.65)$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_o & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_o \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{K}_N & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_N \end{bmatrix} \quad (2.66)$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} -\mathbf{D}_o & -\mathbf{M}_o \\ -\mathbf{M}_o & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} -\mathbf{D}_N & -\mathbf{M}_N \\ -\mathbf{M}_N & \mathbf{0} \end{bmatrix} \quad (2.67)$$

Subscripts o and N indicate that the quantities subscripted belong either to the *original* or *new* system after a discrete transformation has taken place. Subscripts L and R distinguish between left eigenvectors (or left transformations) and right eigenvectors (or right transformations). If the inverses of the $(2n \times 2n)$ transformation matrices exist; The matrices on the right hand sides of equations (2.65)-(2.67) are the LAMs of the new system $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ and it is clear that the corresponding LAMs

for the original system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ appear on the left hand side. These equations can be written more compactly as

$$\underline{\mathbf{T}}_L^T \underline{\mathbf{M}}_o \underline{\mathbf{T}}_R = \underline{\mathbf{M}}_N \quad (2.68)$$

$$\underline{\mathbf{T}}_L^T \underline{\mathbf{D}}_o \underline{\mathbf{T}}_R = \underline{\mathbf{D}}_N \quad (2.69)$$

$$\underline{\mathbf{T}}_L^T \underline{\mathbf{K}}_o \underline{\mathbf{T}}_R = \underline{\mathbf{K}}_N \quad (2.70)$$

The underlining is used to indicate a quantity of “double-dimension” i.e. matrices of dimension $(2n \times 2n)$. The notation for the *LAMs* is chosen deliberately. $\{\underline{\mathbf{T}}_R, \underline{\mathbf{T}}_L\}$ define the left and right transformation matrices. Matrices $\{\underline{\mathbf{M}}_o, \underline{\mathbf{D}}_o, \underline{\mathbf{K}}_o\}$ are the *LAMs* of the system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ in recognition of the fact that Lancaster published these matrices for scalar polynomials in 1961 [48].

The naming convention derives from the fact that $\underline{\mathbf{M}}_o$ does *not* contain \mathbf{M}_o as a partition, $\underline{\mathbf{D}}_o$ does *not* contain \mathbf{D}_o as a partition and $\underline{\mathbf{K}}_o$ does *not* contain \mathbf{K}_o as a partition. Correspondingly, matrices $\{\underline{\mathbf{M}}_N, \underline{\mathbf{D}}_N, \underline{\mathbf{K}}_N\}$ are the *LAMs* of the system $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ and the major $(n \times n)$ partitions of these *LAMs* comprise $\{\mathbf{0}, \mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$. The general SPT described in equations (2.65)-(2.67) can be said to be a structure preserving equivalence (SPE) if both of $\{\underline{\mathbf{T}}_R, \underline{\mathbf{T}}_L\}$ are invertible.

2.6.1 Diagonalising Structure Preserving Equivalences (DSPEs)

Diagonalising SPEs are required to decouple the original equations of motion such that the new system matrices themselves are diagonal. Matrices $\{\underline{\mathbf{T}}_R, \underline{\mathbf{T}}_L\}$ describe a *diagonalising SPT* for the system $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ if

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_D \\ \mathbf{K}_D & \mathbf{D}_D \end{bmatrix} \quad (2.71)$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{K}_D & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_D \end{bmatrix} \quad (2.72)$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} -\mathbf{D}_D & -\mathbf{M}_D \\ -\mathbf{M}_D & \mathbf{0} \end{bmatrix} \quad (2.73)$$

where $\{\underline{\mathbf{K}}_D, \underline{\mathbf{D}}_D, \underline{\mathbf{M}}_D\}$ are block diagonal matrices. The new system matrices $\{\underline{\mathbf{K}}_D, \underline{\mathbf{D}}_D, \underline{\mathbf{M}}_D\}$ represent the *LAMs* for the system whose coefficient matrices are the diagonal matrices $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$. Equations (2.71), (2.72) and (2.73) can be written in the same form as equations (2.68), (2.69) and (2.70)

$$\underline{\mathbf{T}}_L^T \underline{\mathbf{M}} \underline{\mathbf{T}}_R = \underline{\mathbf{M}}_D \quad (2.74)$$

$$\underline{\mathbf{T}}_L^T \underline{\mathbf{D}} \underline{\mathbf{T}}_R = \underline{\mathbf{D}}_D \quad (2.75)$$

$$\underline{\mathbf{T}}_L^T \underline{\mathbf{K}} \underline{\mathbf{T}}_R = \underline{\mathbf{K}}_D \quad (2.76)$$

Note that the spectrum (the set of eigenvalues) of the diagonal system must obviously be identical to the spectrum of the original system and that this spectrum is obtainable, one pair of eigenvalues at a time, from $\{\underline{\mathbf{K}}_D, \underline{\mathbf{D}}_D, \underline{\mathbf{M}}_D\}$.

$$\begin{aligned}
\mathbf{K}_D &= \text{diag}(k_1, k_2, \dots, k_n) \\
\mathbf{D}_D &= \text{diag}(d_1, d_2, \dots, d_n) \\
\mathbf{M}_D &= \text{diag}(m_1, m_2, \dots, m_n)
\end{aligned} \tag{2.77}$$

If the left and right diagonalising transformations $\{\underline{\mathbf{T}}_L, \underline{\mathbf{T}}_R\}$ are normalised in a particular way, then equation (2.77) can be written as

$$\begin{aligned}
\mathbf{K}_D &= \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \\
\mathbf{D}_D &= \text{diag}(2\zeta_1\lambda_1, 2\zeta_2\lambda_2, \dots, 2\zeta_n\lambda_n) \\
\mathbf{M}_D &= \text{Identity}
\end{aligned} \tag{2.78}$$

The transformation matrices $\{\underline{\mathbf{T}}_R, \underline{\mathbf{T}}_L\}$ are not unique and can be expressed in several different parameterisations, but the most useful for present purposes is:

$$\underline{\mathbf{T}}_L = \begin{bmatrix} \mathbf{F}_L - \frac{1}{2}\mathbf{G}_L\mathbf{D}_D & -\mathbf{G}_L\mathbf{M}_D \\ (\mathbf{G}_L\mathbf{K}_D) & \mathbf{F}_L + \frac{1}{2}\mathbf{G}_L\mathbf{D}_D \end{bmatrix} \tag{2.79}$$

$$\underline{\mathbf{T}}_R = \begin{bmatrix} \mathbf{F}_R - \frac{1}{2}\mathbf{G}_R\mathbf{D}_D & -\mathbf{G}_R\mathbf{M}_D \\ \mathbf{G}_R\mathbf{K}_D & \mathbf{F}_R + \frac{1}{2}\mathbf{G}_R\mathbf{D}_D \end{bmatrix} \tag{2.80}$$

where $\{\mathbf{F}_R, \mathbf{G}_R, \mathbf{G}_L, \mathbf{F}_L\}$ are real-valued $(n \times n)$ matrices satisfying the constraint:

$$\mathbf{F}_R\mathbf{G}_L^T + \mathbf{G}_R\mathbf{F}_L^T = \mathbf{0} \tag{2.81}$$

The matrices $\{\underline{\mathbf{T}}_R, \underline{\mathbf{T}}_L\}$ contain modal information (corresponding to *right* and *left* eigenvectors respectively) in the sense that they can be derived quite directly from these eigenvectors. It follows that matrices $\{\mathbf{F}_R, \mathbf{G}_R\}$ contain the right modal eigenvector information and the matrices $\{\mathbf{F}_L, \mathbf{G}_L\}$ contain the left modal eigenvector information.

Garvey *et al.* [47] presented real-valued transformations which can be derived from the complex modal transformations in which the diagonalised system matrices

$\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$ are real and diagonal. See Appendix A for further details to prove equations (2.79) and (2.80).

2.7 Model Reduction and Mode Expansion Methods

Model reduction methods are techniques which are applied to large finite element models to give faster computation of the natural frequencies and mode shapes of a structure [49]. These techniques are used to reduce the number of degrees of freedom in a structural model. The reduced models (mass, stiffness and damping matrices) may be used when comparing the analytical and experimental models. The transformation matrix in the model reduction, whereby the measured modal vectors are expanded back to the full set of analytical degrees of freedom of the system, may be required to perform modal correlation between analytical and experimental modal vectors or model updating [18, 19]. Figure 2-6 shows system coordinates.

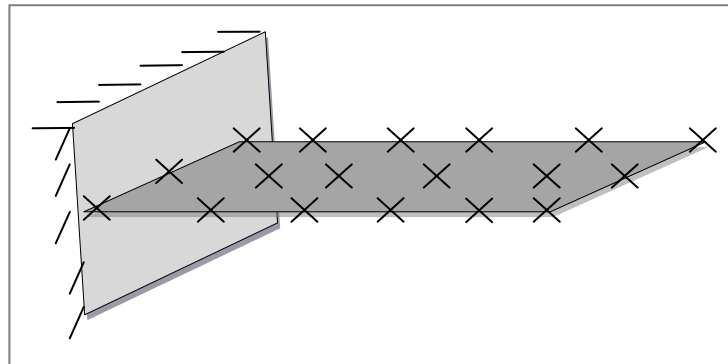


Figure 2-6 : System coordinates

2.7.1 Static Reduction Methods

The oldest model reduction method is that proposed almost simultaneously by Guyan and Irons [23, 24]. The static reduction transformation is based on neglecting the inertia forces for the eliminated (slave) coordinates. The system matrices are divided into partitions which represent the reduced (master) coordinates and the eliminated coordinates. Figure 2-7 shows master and slave coordinates. The master coordinates are coloured red and the eliminated coordinates are coloured black. The eliminated coordinates should be selected as those for which the inertia is low and the stiffness is high. The static reduction method is exact only at zero frequency.

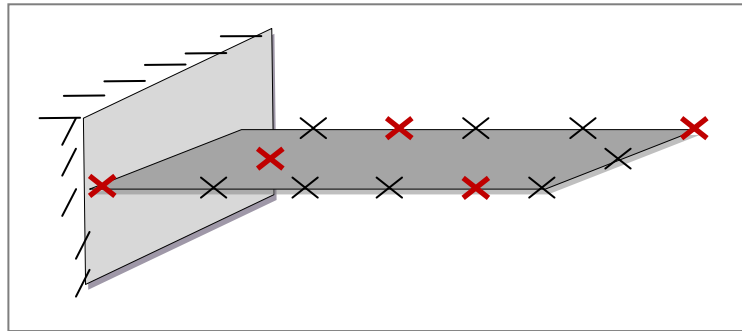


Figure 2-7 : Master and slave coordinates

All elements of the original stiffness matrix contribute to the reduced stiffness matrix but the reduced mass matrix is a combination of stiffness and mass elements. The transformation matrix is based only on the stiffness matrix. The full static system is described by a stiffness matrix and distributed force as

$$\mathbf{K}\mathbf{q} = \mathbf{f} \quad (2.82)$$

where \mathbf{K} is the full system stiffness matrix, \mathbf{f} is the complete force vector acting on the system and \mathbf{q} is the system of displacements for the solution of equation (2.82).

Partitioning equation (2.82) into reduced and eliminated degrees of freedom yields

$$\begin{Bmatrix} \mathbf{f}_r \\ \mathbf{f}_e \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{rr} & \mathbf{K}_{re} \\ \mathbf{K}_{er} & \mathbf{K}_{ee} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_r \\ \mathbf{q}_e \end{Bmatrix} \quad (2.83)$$

where the subscripts ‘ r ’ and ‘ e ’ represent reduced Dofs which are remaining and eliminated Dofs respectively. Taking $\mathbf{f}_e = 0$, find that

$$\mathbf{q}_e = -\mathbf{K}_{ee}^{-1} \mathbf{K}_{er} \mathbf{q}_r \quad (2.84)$$

The coordinate transformation is

$$\mathbf{q} = \begin{Bmatrix} \mathbf{q}_r \\ \mathbf{q}_e \end{Bmatrix} = \mathbf{T}_s \mathbf{q}_r \quad (2.85)$$

The static transformation matrix \mathbf{T}_s can be written as

$$\mathbf{T}_s = \begin{bmatrix} \mathbf{I} \\ \mathbf{t}_s \end{bmatrix} \quad (2.86)$$

where

$$\mathbf{t}_s = -\mathbf{K}_{ee}^{-1} \mathbf{K}_{er} \quad (2.87)$$

The reduced stiffness is

$$\mathbf{K}_r = \mathbf{T}_s^T \mathbf{K} \mathbf{T}_s = \left(\mathbf{K}_{rr} - \mathbf{K}_{re} \mathbf{K}_{ee}^{-1} \mathbf{K}_{er} \right) \quad (2.88)$$

Equation (2.85) is used to transfer the strain energy of the full system into the reduced state producing \mathbf{K}_r described in equation (2.88). The transformation matrix in equation (2.86) can be used to modify the system kinetic energy and generates the reduced mass matrix without considering the equations of motion of the system

$$\mathbf{M}_r = \mathbf{T}_s^T \mathbf{M} \mathbf{T}_s = \left(\mathbf{M}_{rr} - \mathbf{M}_{re} \mathbf{K}_{ee}^{-1} \mathbf{K}_{er} \right) - \left(\mathbf{K}_{ee}^{-1} \mathbf{K}_{er} \right)^T \left(\mathbf{M}_{er} - \mathbf{M}_{ee} \mathbf{K}_{ee}^{-1} \mathbf{K}_{er} \right) \quad (2.89)$$

Garvey *et al.* [43] proposed a method based on a logical extension of the Guyan reduction method for general linear second order systems having damping terms. This method is described here. The left and right static reduction transformation matrices $\{\mathbf{T}_L, \mathbf{T}_R\}$ as in equation (2.86) are extended in two stages.

$$\mathbf{T}_L = \mathbf{T}_{L1} \mathbf{T}_{L2} \quad , \quad \mathbf{T}_R = \mathbf{T}_{R1} \mathbf{T}_{R2} \quad (2.90)$$

The first transformations $\{\mathbf{T}_{L1}, \mathbf{T}_{R1}\}$ are square matrices which decouple the eliminated degrees of freedom from the reduced degrees of freedom in the stiffness matrix.

$$\mathbf{T}_{L1} = \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{t}_L & \mathbf{I} \end{bmatrix} \quad , \quad \mathbf{T}_{R1} = \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{t}_R & \mathbf{I} \end{bmatrix} \quad (2.91)$$

The $\{\mathbf{t}_L, \mathbf{t}_R\}$ represent Guyan reduction transformations which are described in section 2.7.1 equation (2.87). The second transformations $\{\mathbf{T}_{L2}, \mathbf{T}_{R2}\}$ decouple the eliminated coordinates from the reduced coordinates

$$\mathbf{T}_{L2} = \begin{bmatrix} \mathbf{I} \\ 0 \end{bmatrix} \quad , \quad \mathbf{T}_{R2} = \begin{bmatrix} \mathbf{I} \\ 0 \end{bmatrix} \quad (2.92)$$

2.7.2 *System Equivalent Reduction Expansion Process*

O'Callahan [50] introduced another model reduction technique which he called the system equivalent reduction/expansion process (SEREP). This technique produces reduced models and preserves the dynamic character of the original full system models for the selected modes of interest. The accuracy of the SEREP reduced models is not dependent on which degrees of freedom are selected as slaves or masters. The expansion of the reduced system's mode shapes are exactly the same as the original mode shapes of the full system model Φ . The eigensolution of the undamped system is

$$\mathbf{q} = \Phi \mathbf{p} \quad (2.93)$$

Φ is the $(n \times m)$ full system's modal matrix whose columns are m modal vectors and \mathbf{p} is the $(m \times 1)$ displacement vector in the modal coordinate system. The full system is partitioned into retained degrees of freedom and eliminated degrees of freedom.

$$\mathbf{q} = \begin{bmatrix} \Phi_r \\ \Phi_e \end{bmatrix} \mathbf{p} \quad (2.94)$$

Considering only the retained degrees of freedom in equation (2.94) yields

$$\mathbf{q}_r = \Phi_r \mathbf{p} \quad (2.95)$$

The generalized left inverse of Φ_r is required to solve \mathbf{p} . The number of reduced degrees of freedom ' r ' is greater than or equal to the number of modes m in most practical applications used. The formulation of the generalized left inverse of Φ_r is

$$\Phi_r^+ = (\Phi_r^T \Phi_r)^{-1} \Phi_r^T \quad (2.96)$$

The full displacement vector in terms of the reduced system's displacement vector is

$$\mathbf{q} = \Phi \Phi_r^+ \mathbf{q}_r \quad (2.97)$$

The transformation matrix \mathbf{T}_{serep} which relates the reduced degrees of freedom to the full degrees of freedom is

$$\mathbf{T}_{serep} = \Phi \Phi_r^+ = \begin{bmatrix} \Phi_r \Phi_r^+ \\ \Phi_e \Phi_r^+ \end{bmatrix} \quad (2.98)$$

The reduction/expansion process is reversible. The modal vector matrix of the full degrees of freedom system is

$$\Phi = \mathbf{T}_{serep} \Phi_r^r \quad (2.99)$$

whereas $\Phi_r = \Phi_r^r$ and Φ_r^r is the modal matrix formed from the eigensolution of the reduced degrees of freedom system.

2.7.3 Improved Reduction System

The Improved Reduced System (IRS) method was proposed by O’Callahan [51]. This technique comprises an extra term which is added to the static transformation to make some allowance for the inertia forces. The IRS produces an estimate of the reduced eigensystem starting from the Guyan/Irons [52, 53] reduction technique. The modal vectors of the estimated solution can be adjusted as in the static solution producing an improved set of eigenvectors. Reduced mass and stiffness matrices can be produced, and the improved eigensystem is more accurate than that obtained with other techniques [51].

Expanding the upper and lower partitions of equation (2.83) produces the reduced set of equations and the eliminated set of equations

$$\mathbf{q}_e = -\mathbf{K}_{ee}^{-1} \mathbf{K}_{er} \mathbf{q}_r + \mathbf{K}_{ee}^{-1} \mathbf{f}_e \quad (2.100)$$

Substituting equation (2.100) into (2.85) produces system transformation matrix \mathbf{q}

$$\mathbf{q} = \begin{Bmatrix} \mathbf{q}_r \\ \mathbf{q}_e \end{Bmatrix} = \begin{Bmatrix} \mathbf{I} \\ -\mathbf{K}_{ee}^{-1} \mathbf{K}_{er} \end{Bmatrix} \mathbf{q}_r + \begin{Bmatrix} \mathbf{0} \\ \mathbf{K}_{ee}^{-1} \mathbf{f}_e \end{Bmatrix} = \mathbf{T}_s \mathbf{q}_r + \mathbf{X}^{Fe} \quad (2.101)$$

in which \mathbf{X}^{Fe} is the eliminated distributed force

$$\mathbf{X}^{Fe} = \mathbf{Q}_e \mathbf{f} \quad (2.102)$$

and \mathbf{Q}_e is a flexibility matrix associated with the eliminated force of the system

$$\mathbf{Q}_e = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{K}_{ee}^{-1} \end{bmatrix} \quad (2.103)$$

The eigenpairs of the reduced system using Guyan/Irons reduction technique equation (2.85) can be described as

$$\mathbf{K}_r \Phi_r = \mathbf{M}_r \Phi_r \omega^2 \quad (2.104)$$

The full space modal vectors Φ^f can be formed using the static expansion matrix

$$\Phi^f = \mathbf{T}_s \Phi_r \quad (2.105)$$

The force approximation for the distributed inertia forces associated with the eliminated degrees of freedom can be written as

$$\mathbf{K}\Phi^{f'} = \mathbf{f}^f = \mathbf{M}\Phi^f \omega^2 \quad (2.106)$$

Equation (2.106) allows the adjustment of the inertia terms associated with the eliminated degrees of freedom using equations (2.102) and (2.105)

$$\mathbf{X}^{Fe} = \mathbf{Q}_e \mathbf{f}^f = \mathbf{Q}_e \mathbf{M} \Phi^f \omega^2 = \mathbf{Q}_e \mathbf{M} \mathbf{T}_s \Phi_r \omega^2 \quad (2.107)$$

The improved modal vectors which include the inertia forces can be formed using equations (2.101), (2.104), (2.105) and (2.107)

$$\Phi_{irs} = \Phi^f + \mathbf{X}^{Fe} = \mathbf{T}_s \Phi_r + \mathbf{Q}_e \mathbf{M} \mathbf{T}_s \mathbf{M}_r^{-1} \mathbf{K}_r \Phi_r \quad (2.108)$$

The improved transformation matrix \mathbf{T}_{irs} can be written as

$$\mathbf{T}_{irs} = \mathbf{T}_s + \mathbf{Q}_e \mathbf{M} \mathbf{T}_s \mathbf{M}_r^{-1} \mathbf{K}_r = \begin{bmatrix} \mathbf{I} \\ \mathbf{t}_s + \mathbf{t}_e \end{bmatrix} \quad (2.109)$$

where \mathbf{t}_e is defined as

$$\mathbf{t}_e = \left(\mathbf{K}_{ee}^{-1} \mathbf{M}_{er} + \mathbf{K}_{ee}^{-1} \mathbf{M}_{ee} \mathbf{t}_s \right) \mathbf{M}_r^{-1} \mathbf{K}_r \quad (2.110)$$

The improved transformation produces an improved relationship between eliminated coordinates and reduced coordinates, also is used to expand the reduced degrees of freedom to the full system degrees of freedom. Equation (2.100) becomes

$$\mathbf{q}_e = \left(-\mathbf{K}_{ee}^{-1} \mathbf{K}_{er} + \mathbf{K}_{ee}^{-1} \left(\mathbf{M}_{er} - \mathbf{M}_{ee} \mathbf{K}_{ee}^{-1} \mathbf{K}_{er} \right) \mathbf{M}_r^{-1} \mathbf{K}_r \right) \mathbf{q}_r \quad (2.111)$$

2.7.4 Dynamic Reduction Method

The reduction of static problems is exact when using a static reduction method proposed by Guyan and Irons [52, 53]. The static transformation matrix that is used to reduce the system mass and stiffness matrices neglects the inertia forces for the eliminated coordinates. Many errors are produced when the static reduction method is applied to the reduction of dynamic systems. Several alternatives [54-63] have been proposed to modify the Guyan reduction method to decrease these errors. The different methods based on a truncated series expansion depend on the convergence of the series. These are proposed to improve the eigenvectors and not eigenvalues [64].

The dynamic reduction transformation does take account of inertia forces for the eliminated degrees of freedom. The transformation for the dynamic reduction method is accurate at one specific frequency. Miller *et al.* [65] examined a dynamic reduction process to investigate the errors which may be introduced in the eigenvalues and eigenvectors.

The equation of motion describing free vibration of the system is partitioned as

$$\begin{bmatrix} \mathbf{M}_{rr} & \mathbf{M}_{re} \\ \mathbf{M}_{er} & \mathbf{M}_{ee} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}}_r \\ \ddot{\mathbf{q}}_e \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{rr} & \mathbf{K}_{re} \\ \mathbf{K}_{er} & \mathbf{K}_{ee} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_r \\ \mathbf{q}_e \end{Bmatrix} = 0 \quad (2.112)$$

Then the eigenvalue problem can be written as

$$-\omega^2 \begin{bmatrix} \mathbf{M}_{rr} & \mathbf{M}_{re} \\ \mathbf{M}_{er} & \mathbf{M}_{ee} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_r \\ \mathbf{q}_e \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{rr} & \mathbf{K}_{re} \\ \mathbf{K}_{er} & \mathbf{K}_{ee} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_r \\ \mathbf{q}_e \end{Bmatrix} = 0 \quad (2.113)$$

The response of eliminated degrees of freedom can be estimated from reduced degree of freedom system using the second row of the matrices in equation (2.113)

$$\mathbf{q}_e = -(\mathbf{K}_{ee} - \omega^2 \mathbf{M}_{ee})^{-1} (\mathbf{K}_{er} - \omega^2 \mathbf{M}_{er}) \mathbf{q}_r = \mathbf{t}_d \mathbf{q}_r \quad (2.114)$$

The dynamic reduction transformation is defined as

$$\mathbf{T}_d = \begin{bmatrix} \mathbf{I} \\ \mathbf{t}_d \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ -(\mathbf{K}_{ee} - \omega^2 \mathbf{M}_{ee})^{-1} (\mathbf{K}_{er} - \omega^2 \mathbf{M}_{er}) \end{bmatrix} \quad (2.115)$$

2.7.5 *Iterative of IRS (IIRS)*

The reduced system produced from IRS matches the lower frequency resonances of the full system better than Guyan reduction. On the other hand, the reduced stiffness matrix is stiffer and the reduced mass matrix is less suitable than that produced from Guyan reduction [66]. Gordis [67] developed the transformation for the improved reduced system IRS by using a binomial series expansion. This transformation is exactly correct at a given frequency.

Friswell *et al.* [66, 68, 69] proposed an alternative model reduction process based on the IRS procedure of O'Callahan [51]. The Iterated Improved Reduced System (IIRS) techniques based on the IRS transformation is produced from the dynamic reduction method instead of static reduction. The improved estimate of the reduced matrices is used to define the IRS transformation that for more accurate transformation at each iteration.

The Improved Reduced System (IRS) method has been extended from equation of motion including the inertia term. The eliminated coordinates becomes

$$\mathbf{q}_e = -(\mathbf{K}_{ee} - \Omega^2 \mathbf{M}_{ee})^{-1} (\mathbf{K}_{er} - \Omega^2 \mathbf{M}_{er}) \mathbf{q}_r \quad (2.116)$$

Rearranging the eliminated coordinates in equations (2.116) and (2.114) for both IRS and dynamic reduction and using binomial theorem (in Appendix B) yields

$$\mathbf{q}_e = -\mathbf{C}_{ee}^{-1} \left[\mathbf{C}_{er} + (\Omega^2 - \omega^2) (\mathbf{M}_{ee} \mathbf{C}_{ee}^{-1} \mathbf{C}_{er} - \mathbf{M}_{er}) \right] \mathbf{q}_r \quad (2.117)$$

where $\mathbf{C}_{ee} = (\mathbf{K}_{ee} - \omega^2 \mathbf{M}_{ee})$ and $\mathbf{C}_{er} = (\mathbf{K}_{er} - \omega^2 \mathbf{M}_{er})$. Ω is natural frequency and ω is a given frequency.

The dynamic IRS transformation \mathbf{T}_{dirs} depends on the reduced mass and stiffness matrices obtained from dynamic reduction.

$$\mathbf{T}_{dirs} = \mathbf{T}_d + \mathbf{P}_e \mathbf{M} \mathbf{T}_d \mathbf{M}_r^{-1} \mathbf{C}_r \quad (2.118)$$

in which $\mathbf{C}_r = (\mathbf{K}_r - \omega^2 \mathbf{M}_r)$ and

$$\mathbf{P}_e = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{C}_{ee}^{-1} \end{bmatrix} \quad (2.119)$$

The Iterated Improved Reduced System (IIRS) transformation [66] can be written as

$$\mathbf{T}_{irs,i+1} = \mathbf{T}_s + \mathbf{Q}_e \mathbf{M} \mathbf{T}_{irs,i} \mathbf{M}_{irs,i}^{-1} \mathbf{K}_{irs,i} \quad (2.120)$$

The speed of convergence depends on the choice of the reduced degrees of freedom, and a poor choice leads to a very slow rate of convergence. On convergence the transformation is

$$\mathbf{T}_{irsi} = \mathbf{T}_{irsi+1} = \mathbf{T} \quad (2.121)$$

The iterative algorithm converges a subset of the natural frequencies of the full model. Qu and Fu [70] proposed two iterative methods for the dynamic reduction based on the modified eigenvalue equation. This convergence is higher than the other methods; however, the computational work is more expensive [71].

2.7.6 *Model reduction for damped systems*

Most of the proposed reduction methods are implemented to decouple and reduce out the eliminated degrees of freedom from the full system by selecting one slave degree of freedom at a time. Guyan reduction and the improved transformation are developed for undamped systems and they work extremely well. These methods are also suitable for proportionally damped systems. Qu *et al.* [72] proposed methods for non classically damped systems in displacement space based on static reduction. However, the accuracy of these approaches is poor for heavily damped systems, because the reduction transformation is independent of any damped term.

Most of the dynamic reduction transformation matrices do not comprise a damping term. The iteration methods do not converge to realistic values. Other iterative approaches are proposed by Qu *et al.* [71] for non classically damped systems based on a dynamic reduction which is defined in state space. The accuracy of these methods are much better than that in displacement space. Although the system matrices are defined in state space; an expression of the stiffness, mass and damping matrices in displacement space cannot be found.

Many investigators [73-76] have proposed solutions to this problem for state space systems but all have notable deficiencies. Garvey *et al.* [43] proposed a method based on a logical extension of Guyan reduction method for general linear second order systems having damping terms. The method is developed to perform the model reduction based on the elementary structure preserving coordinate transformations (SPEs) for second order systems. More details can be found in references [43, 44].

2.7.7 Craig-Bampton Reduction Method

Complex structures such as airplane or automotive components require coupling the reduced model systems by dynamic substructuring. Substructuring methods have been used to replace the result of a large eigenvalue problem with several small eigenvalue problems [77].

Component mode methods [78] have been applied to the analysis of large complex structural systems when the response to dynamic excitation must be analysed. Substructuring involves three basic steps: dividing the structure into a number of substructures or components, obtaining sets of “component modes” which are used in the model reduction, and coupling of the component mode models to form a reduced order system model. Component mode synthesis (CMS) methods have been developed for damped systems as well as for undamped systems [79].

An alternative method for model reduction was proposed by Craig & Bampton [80]. The Craig-Bampton transformation is one of the component mode synthesis (CMS) methods. This method is used for reducing the size of a finite element model that combines motion of boundary points with modes of the structure assuming that the boundary points are held fixed.

The method employs two forms of generalized coordinates. Boundary generalized coordinates are related to the displacement modes of the substructures known as constraint modes. All constraint modes are generated by matrix operations from substructure input data. Substructure normal mode generalized coordinates are

related to free vibration modes of the substructures relative to completely restrained boundaries.

2.7.7.1 Craig-Bampton reduction for undamped substructures

The Craig-Bampton reduction [79, 80] is explained here for the general case of undamped natural second order form (N.S.O.F) models. The equation of motion of an undamped substructure is represented as

$$\mathbf{M}\ddot{\mathbf{q}}_s + \mathbf{K}\mathbf{q}_s = \mathbf{f}_s \quad (2.122)$$

equation (2.122) can be expressed in terms of the boundary degrees of freedom and interior degrees of freedom as

$$\begin{bmatrix} \mathbf{M}_{bb} & \mathbf{M}_{bi} \\ \mathbf{M}_{ib} & \mathbf{M}_{ii} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}}_b \\ \ddot{\mathbf{q}}_i \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{bb} & \mathbf{K}_{bi} \\ \mathbf{K}_{ib} & \mathbf{K}_{ii} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_b \\ \mathbf{q}_i \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_b \\ \mathbf{f}_i \end{Bmatrix} \quad (2.123)$$

where the subscripts “*b*” and “*i*” indicate *boundary* and *interior* quantities respectively, \mathbf{q}_b is the vector of boundary degrees of freedom, \mathbf{q}_i is the vector of internal degrees of freedom. Figure 2-8 shows substructuring coordinates. The boundary coordinates are coloured green and the interior coordinates are coloured blue.

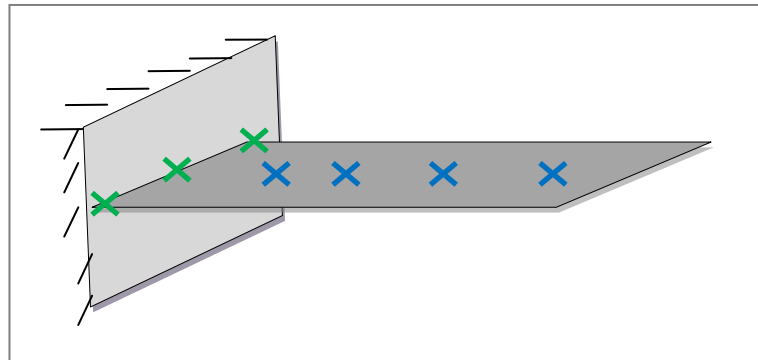


Figure 2-8 : Boundary and interior coordinates

The Craig-Bampton transformation [79] is just like any other coordinate transformation which is defined as

$$\mathbf{q}_s = \mathbf{T}_{cb} \mathbf{q}_n = \mathbf{T}_{cb} \begin{Bmatrix} \mathbf{q}_b \\ \mathbf{q}_i \end{Bmatrix} \quad (2.124)$$

where \mathbf{q}_i is the vector of modal degrees of freedom and \mathbf{T}_{cb} is the Craig-Bampton transformation which is divided into two parts

$$\mathbf{T}_{cb} = [\mathbf{F} \quad \mathbf{E}] \quad (2.125)$$

where

$$\mathbf{F} = \begin{bmatrix} \mathbf{I} \\ \boldsymbol{\Phi}^c \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\Phi}^n \end{bmatrix} \quad (2.126)$$

$\boldsymbol{\Phi}^c$ represents the constraint modes, and $\boldsymbol{\Phi}^n$ represents the normal modes of the constrained substructure. The constraint modes $\boldsymbol{\Phi}^c$ are determined from Guyan reduction, when the forces at all interior degrees of freedom are set equal to zero

$$\boldsymbol{\Phi}^c = -\mathbf{K}_{ii}^{-1} \mathbf{K}_{ib} \quad (2.127)$$

The substructure normal mode $\boldsymbol{\Phi}^n$ is defined as the normal modes of the substructure with totally constrained boundary. The eigenvalue problem of the interior coordinates can be written

$$\mathbf{K}_{ii} \mathbf{q}_i = \omega^2 \mathbf{M}_{ii} \mathbf{q}_i \quad (2.128)$$

The eigenvectors of equation (2.128) represents the normal modes of the constrained substructure (*interior* coordinates).

These eigenvectors form the columns of the matrix $\boldsymbol{\Phi}^n$. The reduced system mass and stiffness matrices are

$$\mathbf{M}_{cb} = \mathbf{T}_{cb}^T \mathbf{M} \mathbf{T}_{cb} = \begin{bmatrix} \mathbf{M}_{bb} & \mathbf{M}_{bi} \\ \mathbf{M}_{ib} & \mathbf{M}_{ii} \end{bmatrix} \quad (2.129)$$

$$\mathbf{K}_{cb} = \mathbf{T}_{cb}^T \mathbf{K} \mathbf{T}_{cb} = \begin{bmatrix} \mathbf{K}_{bb} & \mathbf{K}_{bi} \\ \mathbf{K}_{ib} & \mathbf{K}_{ii} \end{bmatrix} \quad (2.130)$$

If the mode shapes have been normalized, then $\mathbf{M}_{ii} = [\mathbf{I}]$ and $\mathbf{K}_{pp} = [\omega^2]$ are diagonal

matrices. The dynamic equation of motion including damping can be written as

$$\begin{bmatrix} \mathbf{M}_{bb} & \mathbf{M}_{bp} \\ \mathbf{M}_{pb} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}}_b \\ \ddot{\mathbf{q}}_i \end{Bmatrix} + \begin{bmatrix} \mathbf{D}_{bb} & \mathbf{D}_{bp} \\ \mathbf{D}_{pb} & 2\zeta\omega \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{q}}_b \\ \dot{\mathbf{q}}_i \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{bb} & 0 \\ 0 & \omega^2 \end{bmatrix} \begin{Bmatrix} \mathbf{q}_b \\ \mathbf{q}_i \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_b \\ \mathbf{0} \end{Bmatrix} \quad (2.131)$$

Chapter 9 generalises Craig-Bampton transformations for general damped systems without input and output assumptions.

2.8 Force appropriation

2.8.1 Force appropriation methods

Many methods have been developed to determine the appropriated force vector required to excite pure normal modes. The force appropriation techniques are divided into direct and iterative methods [13, 14]. The iterative force appropriation methods are rarely used, as they can be time consuming and difficult to apply [13]. The direct force appropriation methods have three stage. The first stage is to measure the frequency response function (FRF) matrices relating selected response positions to multiple input exciters [13, 14]. The second stage is to estimate the undamped natural frequencies of the normal modes. A monophase vector of excitation forces is derived for each normal mode that will generate a single mode monophase response. The final stage is to apply the force vectors corresponding to each mode and related undamped natural frequency to measure the normal mode shape [13].

For a linear system which is subject to monophas sinusoidal excitation forces at frequency ω , with given force vector \mathbf{u} , the complex displacement response of the structure in steady state is

$$\mathbf{y} = \mathbf{H}(\omega)\mathbf{u} = (\mathbf{A}(\omega) + i\mathbf{B}(\omega))\mathbf{u} \quad (2.132)$$

where \mathbf{y} is the vector of responses, $\mathbf{A}(\omega)$ and $\mathbf{B}(\omega)$ are the real and imaginary parts of the frequency response function (FRF) matrix. The j^{th} undamped normal mode is excited at the corresponding undamped natural frequency ω_j when the response of the structure is in monophas and in quadrature with the excitation. In this condition the real part of the response will be zero while the imaginary part corresponds to the undamped normal mode shape $\{\phi\}_j$.

$$\text{Re}(\mathbf{y}) = \mathbf{A}(\omega)\mathbf{u}_j = 0 \quad (2.133)$$

$$\text{Im}(\mathbf{y}) = \mathbf{B}(\omega)\mathbf{u}_j = \phi_j \quad (2.134)$$

The multivariate mode indicator function (MMIF) is one of the most common methods of force appropriation. This method considers the ratio of the kinetic energy of the response in phase with the excitation to that of the total response. Wright *et al.* [15] noted that good quality force appropriation requires multiple exciters to be positioned. The appropriated force vector derived for a particular normal mode will excite only that mode. For proportionally damped structures, the appropriated force vector must excite the mode of interest and not contribute to the other modes, while for non-proportionally damped systems, the force vector must also cancel the non-proportional damping coupling force between the mode of interest and other modes [13, 15].

2.8.2 *Square FRF matrix method*

For square FRF matrices (number of output measurements equals the number of exciters), there are many methods to find the non-trivial solutions of the homogeneous equations (2.133). Asher's method [81] uses the determinant of the real frequency response function $\mathbf{A}(\omega)$ in equation (2.133) to calculate the natural frequencies.

$$|\mathbf{A}(\omega)| = 0 \quad (2.135)$$

Equation (2.134) is solved directly using the adjoint of $\mathbf{A}(\omega)$ or Gauss-Seidel method. If $\mathbf{A}(\omega)$ is not singular, the force vector \mathbf{u}_j will be trivial. The modified Asher method [82] for an eigenvalue solution can be written as

$$\mathbf{A}(\omega)\mathbf{u}_j = \lambda_j\mathbf{u}_j \quad (2.136)$$

The Trail-Nash method [83] proposed an alternative approach for general eigenvalues

$$\mathbf{A}(\omega)\mathbf{u}_j = \lambda_j\mathbf{B}(\omega)\mathbf{u}_j \quad (2.137)$$

Natural frequencies are identified from zero crossing of the eigenvalues λ . The corresponding eigenvectors give the appropriation force vectors for each mode.

To allow more responses to be measured the number of exciters is increased. The methods fail if this number exceeds the effective number of degrees of freedom and the FRF matrix is rank deficient.

2.8.3 Rectangular FRF matrix method

For rectangular *FRF* matrices [13] (number of output measurements are greater than the number of exciters), the real part of the response is minimised across all of the response measurements. The extended Asher's method minimises the sum of the squares of the real part of the response with respect to the force vector, leading to the eigenvalue problem

$$\text{Re}(\mathbf{H}(\omega))^T \text{Re}(\mathbf{H}(\omega)) \mathbf{u}_j = \lambda_j \mathbf{u}_j \quad (2.138)$$

The multivariate mode indicator function (*MMIF*) minimises the ratio of the sum of the squares of the real part of the response to the sum of the squares of the total response

$$\text{Re}(\mathbf{H}(\omega))^T \text{Re}(\mathbf{H}(\omega)) \mathbf{u}_j = \lambda_j \left(\text{Re}(\mathbf{H}(\omega))^T \text{Re}(\mathbf{H}(\omega)) + \text{Im}(\mathbf{H}(\omega))^T \text{Im}(\mathbf{H}(\omega)) \right) \mathbf{u}_j \quad (2.139)$$

The undamped natural frequencies are identified by minima of the eigenvalues λ .

For rectangular *FRF* matrices a rank reduction technique is employed to reduce the size of the eigenvalue problem and so generate principle force vectors. The first singular value decomposition (SVD) [84] of $\text{Re}(\mathbf{H}(\omega))$ is carried out and the decomposition partitioned according to the effective rank of $\text{Re}(\mathbf{H}(\omega))$ at a given resonance. The second SVD yields the appropriated force vectors.

$$[\mathbf{U} \quad \Sigma \quad \mathbf{V}] = \text{svd}(\text{Re}(\mathbf{H}(\omega))) \quad (2.140)$$

$$\text{Re}(\mathbf{H}(\omega)) = \mathbf{U} \Sigma \mathbf{V}^T \quad (2.141)$$

The singular values Σ are used to identify the undamped natural frequencies.

2.9 Modal Correlation Methods

Modal correlation is well-known as a technique is performed to combine and quantify two sets of data [43]. The correlation of experimental and theoretical results is used to identify any discrepancies between their properties. These measures have been developed for undamped or classically damped systems. The major rationale for modal correlation is to quantify and evaluate the uniformity between modal vectors from the same system. Modal correlation also provides a method to quantify the discrepancy between predicted and physical modal vectors [23, 85]. The basic requirement for all modal correlation measures [85] is

$$\text{MAC} = \frac{\|\phi_i^T \mathbf{X} \phi_j\|^2}{|\phi_i^T \mathbf{X} \phi_i| |\phi_j^T \mathbf{X} \phi_j|} \quad (2.142)$$

where ϕ_i, ϕ_j are two different modal vectors. \mathbf{X} is a scaling matrix.

Although several methods have been proposed in the literature to assess these discrepancies, two techniques [86] are most commonly known and used for the determination of modal vector correlation. The first technique is the modal assurance criterion (MAC), this has become popular for correlating measured modes with predicted modes. The second technique is based on the orthogonality check that uses the analytical and experimental vectors with the analytical mass matrix.

Modal correlation methods can be categorised in different ways. The most important division between classes is whether the correlation is performed at the level of full length calculated modal vectors or directly on the level of the measured modal vectors [23, 86]. If the assessment is to be carried out using the full vectors,

expanding the measured modal to at level of the full length analytical modal is possible by using one of the modal/mode expansion techniques. However, if the measure of modal correlation is performed on the measured modal vector, the reduced modal vectors must be provided to match the experimental modal vectors where model reduction techniques are used.

2.9.1 The Modal Assurance Criterion (MAC)

The modal assurance criterion [85] is the most common measure of modal correlation used. It was defined initially for real valued modes and subsequently extended to complex modes [23, 87]. The development of the modal assurance criteria over the last twenty years has led to similar assurance criteria being used for experimental and analytical structural dynamics such as coordinate modal assurance criterion (COMAC), frequency response assurance criterion (FRAC), coordinate orthogonality check (COOC), frequency scaled modal assurance criterion (FSMAC), partial modal assurance criterion (PMAC), scaled modal assurance criterion (SMAC), and modal assurance criterion using reciprocal modal vectors [88].

The MAC has been used to estimate the degree of correlation between two sets of modes, and to equate the measured modes with the computed modes which are reduced to match the measurement locations [85, 86]. The major advantage of modal assurance criterion is that no complete experimental eigenvectors and no mass matrices are required in order to assess the correlation between different modal vectors [85-87]. The modal assurance criterion is a statistical indicator, just like ordinary coherence [88]. The MAC satisfies the basic requirement for all modal

correlation measures independent of scaling in equation (2.142) where \mathbf{X} is the identity matrix.

This technique has limitations based on its formulation [87]. The MAC value is between zero and one. The modal vectors are mismatched if MAC indicates nearly zeros. This can be due to some reasons such as a change in the system matrices during modal testing, nonlinearity in the system generated from different exciter positions or excitation signals, noise associated with modal vectors, and so on. Modal parameters may exploit a complex system model when a real valued system model exists or the modal vectors are from different excitation positions. The MAC can indicate consistency if it's value is near unity.

The modal assurance criterion was developed and used for vector correlation in experimental data and moreover, to compare which vectors are similar and completely different to each other [89]. These vectors are obtained from a single row, column, several rows or columns of the transfer function matrix [86]. Allemang and Brown [85] focused on the development of the concept of consistency of modal vectors evaluated through the use of the modal assurance criterion (MAC) and the modal scale factor (MSF).

The modal scale factor [85] is used to provide a means of normalizing all estimates of the same modal vector. The modal assurance criterion provides a measure of the consistency between different modal vectors for the same system. Both modal assurance criterion and modal scale factor provide a method of comparing of modal

vectors originating from different sources. The transfer function can be written in the form

$$\mathbf{H}(s) = \frac{\mathbf{y}(s)}{\mathbf{u}(s)} = \sum_{r=1}^n \frac{\mathbf{A}(r)}{s - \mathbf{p}(r)} \quad (2.143)$$

where $\mathbf{y}(s)$ is the response vector, $\mathbf{H}(s)$ is the transfer function, $\mathbf{u}(s)$ is the excitation vector, s is the Laplace variable $j\omega$, $\mathbf{A}(r)$ is the residue matrix, r is the mode number, n is the number of degrees of freedom, and $\mathbf{p}(r)$ is system pole for mode r . Each element of the residue matrix can be written as

$$a(i, j, r) = k(r) \phi(i, r) \phi(j, r) \quad (2.144)$$

in which $k(r)$ is the scaling constant for mode r , $\phi(i, r)$ is the modal coefficient for location i of mode r , and $\phi(j, r)$ is the modal coefficient for location j of mode r . The modal coefficient of the excitation location and the modal coefficient of the response location can be written as

$$\mathbf{A}(r) = k(r) \phi(r) \phi(r)^T \quad (2.145)$$

Equation (2.145) shows that each row or column of $\{\phi(r) \phi(r)^T\}$ is the same modal vector multiplied by the modal coefficient of the response location or excitation location. In other word, each element of the residue matrix $\mathbf{A}(r)$ [85] consists the product of scaling constant.

$$\mathbf{A}(r) = k(r) \begin{bmatrix} \phi(1, r) \phi(1, r) & \phi(1, r) \phi(2, r) & \phi(1, r) \phi(3, r) & \dots \\ \phi(2, r) \phi(1, r) & \phi(2, r) \phi(2, r) & \phi(2, r) \phi(3, r) & \dots \\ \phi(3, r) \phi(1, r) & \phi(3, r) \phi(2, r) & \phi(3, r) \phi(3, r) & \dots \\ \dots & \dots & \dots & \dots \\ \phi(m, r) \phi(1, r) & \phi(m, r) \phi(2, r) & \phi(m, r) \phi(3, r) & \dots \end{bmatrix} \quad (2.146)$$

The ratio between rows c, d or columns c, d is constant. The modal scale factor is equal to:

$$\mathbf{MSF}(c, d, r) = \frac{\phi(c, r)}{\phi(d, r)} \quad (2.147)$$

The above equation is used to calculate a least squares error estimate of the proportionality constant between rows or columns of the residue matrix, the model is linear.

$$\mathbf{A}(c, r) = \mathbf{MSF}(c, d) \mathbf{A}(d, r) \quad (2.148)$$

The value of the modal scale factor is to be calculated so as to minimize the sum of the squared errors between corresponding elements of each modal vector. The modal scale factor is

$$\mathbf{MSF}(c, d) = \frac{\mathbf{MOM}(c, d)}{\mathbf{MOM}(d, d)} = \frac{\sum_{j=1}^n a(c, j, r) a^*(d, j, r)}{\sum_{j=1}^n a(d, j, r) a^*(d, j, r)} \quad (2.149)$$

The numerator of the above equation can be defined as the cross moment of the modal vectors. The denominator of the above equation can be defined as the auto moment of the modal vectors.

The modal assurance criterion is defined as a scalar constant relating the portion of the auto moment of the modal vector.

$$\mathbf{MAC}(c, d) = \frac{|\mathbf{MSF}(c, d)|^2 \mathbf{MOM}(d, d)}{\mathbf{MOM}(c, c)} \quad (2.150)$$

Substituting equation (2.149) into equation (2.150) yields

$$\mathbf{MAC}(c, d) = \frac{|\mathbf{MOM}(c, d)|^2}{\mathbf{MOM}(c, c) \mathbf{MOM}(d, d)} \quad (2.151)$$

2.9.2 Cross Orthogonality Checks (XOC)

Cross orthogonality checks [86] are used to check whether each modal vector of a system is orthonormal to all other modal vectors of that system. Modal correlation based on orthogonality relationships are commonly used to form normalised modal vectors using the analytical mass matrix [85].

$$\phi_i^T \mathbf{M} \phi_j = 0 \quad i \neq j \quad (2.152)$$

$$\phi_i^T \mathbf{M} \phi_j = \mathbf{M}_i \quad i = j \quad (2.153)$$

where \mathbf{M}_i is the generalised mass of mode i .

The acceptable criterion from equation (2.153) is for all generalised masses to be unity and all terms from equation (2.152) to be satisfied. In practice, equation (2.152) might not be achieved for some reason; for example, lack of measured modal vectors, the mass matrix is not related to the physical system or the reduced mass matrix is invalid. This technique has limitations based on its formulation.

Avitabile *et al.* [86] proposed a technique which is called a pseudo orthogonality check to overcome some of the problems associated with the modal assurance criteria and orthogonality check. This technique uses a system equivalent reduction/expansion process (SEREP) to reduce the mass matrix to the set of test degrees of freedom and the effects of the reduction are minimized.

The same reduction technique develops a transformation matrix that can be used to expand the measured modal vectors back to the full set of analytical degrees of

freedom of the system. The pseudo orthogonality check can be performed at the reduced set of measured degrees of freedom or at the full set of computed degrees of freedom of the system.

The transformation \mathbf{T}_{serep} matrix in equation (2.98) is used to form the reduced mass \mathbf{M}_r and stiffness \mathbf{K}_r matrices. The *SEREP* process gives a basis of mapping the reduced set of degrees of freedom to the full set of degrees of freedom as in equation (2.99). All measured mode shapes at the reduced set of test degrees of freedom Φ_m can be expanded to the full set of analytical Φ_m^F .

The analytical system matrices can be improved using the measured modal vectors. Mass is assumed the reference for this. The system mass matrix can be optimized at the full or reduced set of degrees of freedom. At the reduced set of test degrees of freedom, the improved mass matrix [86] can be defined as

$$\mathbf{M}_r^I = \mathbf{M}_r + \Phi_r^{+T} \left[\mathbf{I} - \Phi_m^T \mathbf{M}_r \Phi_m \right] \Phi_r^+ \quad (2.154)$$

Φ_r^+ is the generalised inverse which is defined as

$$\Phi_r^+ = \left(\Phi_m^T \mathbf{M}_r \Phi_r \right)^{-1} \Phi_m^T \mathbf{M}_r \quad (2.155)$$

At the full set of analytical degrees of freedom the improved mass matrix can be defined as

$$\mathbf{M}^I = \mathbf{M} + \Phi^{+T} \left[\mathbf{I} - \Phi_m^{FT} \mathbf{M} \Phi_m^F \right] \Phi^+ \quad (2.156)$$

Φ^+ is the generalised inverse which is defined as

$$\Phi^+ = \left(\Phi_m^{FT} \mathbf{M} \Phi_m^F \right)^{-1} \Phi_m^{FT} \mathbf{M} \quad (2.157)$$

Pseudo orthogonality checks can be investigated at the reduced set of test degrees of freedom or at the full set of analytical degrees of freedom using the analytical mass matrix or the improved mass matrix. The four pseudo orthogonality checks can be made

$$\begin{aligned} \mathbf{PS}_1 &= \mathbf{\Phi}_m^{FT} (\mathbf{M}\mathbf{\Phi}) \quad , \quad \mathbf{PS}_2 = (\mathbf{\Phi}_m^{FT} \mathbf{M}^I) \mathbf{\Phi} \\ \mathbf{PS}_3 &= \tilde{\mathbf{\Phi}}_m^{FT} (\mathbf{M}\mathbf{\Phi}) \quad , \quad \mathbf{PS}_4 = (\mathbf{\Phi}_m^{FT} \mathbf{M}^I) \tilde{\mathbf{\Phi}} \end{aligned} \quad (2.158)$$

where $\tilde{\mathbf{\Phi}}_m^F = \mathbf{\Phi}_m^F (\mathbf{\Phi}_m^{FT} \mathbf{M} \mathbf{\Phi}_m^F)^{-1/2}$ and $\tilde{\mathbf{\Phi}} = \mathbf{\Phi} (\mathbf{\Phi}^T \mathbf{M}^I \mathbf{\Phi})^{-1/2}$ are the normalised set of experimental vectors and normalised set of analytical vectors respectively.

Morales [87] introduced some comments on the modal assurance criterion and the normalized cross orthogonality and a linear modal correlation coefficient. Normalized cross orthogonality is a modification of the *MAC* adapted to solve its defects. It is an orthogonality check. The other important implication is that the normalized cross orthogonality limit has a meaningful domain. The result of the Linear normalized cross orthogonality indicates that the situation is characterized by the correlation.

2.9.3 *Modal correlation for damped system*

Mass weighted cross orthogonality check is commonly used for classically damped systems. The mass cross orthogonality measure of modal correlation is insufficient to detect all discrepancies between physical and analytical modal information [23]. Even a full set of full length measured modes is presented; modal reduction method does not produce a transformation matrix spanning the space of the computed modal

vectors. For an undamped system, the mass weighted cross orthogonality measure of modal correlation [23] can be written as

$$\mathbf{XO}_M = \Phi_a^T \mathbf{M} \Phi_m \quad (2.159)$$

The measured characteristic root information can be included in the stiffness weighted cross orthogonality measure of modal correlation [23] can be expressed as

$$\mathbf{XO}_K = \Lambda_a^{-1} \Phi_a^T \mathbf{K} \Phi_m \Lambda_m^{-1} \quad (2.160)$$

Most of the demonstrated modal correlation methods do not consider the eigenvalues (frequencies) associated with eigenvectors (modes). Garvey *et al.* [23] introduced modal correlation measures for general viscous damped structures. For general viscous damping, the three different cross orthogonality measures of modal correlation can be expressed in terms of the structure preserving matrices as

$$\mathbf{XO}_{KD} = \begin{bmatrix} \mathbf{0} & \Lambda_a \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{W}_a & \mathbf{X}_a \\ \mathbf{Y}_a & \mathbf{Z}_a \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{W}_m & \mathbf{X}_m \\ \mathbf{Y}_m & \mathbf{Z}_m \end{bmatrix} \begin{bmatrix} \mathbf{0} & \Lambda_m \end{bmatrix}^{-1} \quad (2.161)$$

$$\mathbf{XO}_{KM} = \frac{1}{2} \begin{bmatrix} \Lambda_a & \Lambda_a \\ \mathbf{I} & -\mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{W}_a & \mathbf{X}_a \\ \mathbf{Y}_a & \mathbf{Z}_a \end{bmatrix}^T \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{W}_m & \mathbf{X}_m \\ \mathbf{Y}_m & \mathbf{Z}_m \end{bmatrix} \begin{bmatrix} \Lambda_m & \mathbf{I} \\ \Lambda_m & -\mathbf{I} \end{bmatrix}^{-1} \quad (2.162)$$

$$\mathbf{XO}_{DM} = \begin{bmatrix} \zeta_a \Lambda_a & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{W}_a & \mathbf{X}_a \\ \mathbf{Y}_a & \mathbf{Z}_a \end{bmatrix}^T \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W}_m & \mathbf{X}_m \\ \mathbf{Y}_m & \mathbf{Z}_m \end{bmatrix} \begin{bmatrix} \zeta_m \Lambda_m & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}^{-1} \quad (2.163)$$

where $\{\Lambda_a, \Lambda_m\}$ are the computed and measured eigenvalues respectively.

2.10 Model Updating

The procedures of correcting or adjusting the theoretical model to match the modal properties with the experimental results are known as model updating [18, 22, 90].

The model updating techniques are largely used in civil and mechanical engineering to adjust selected parameters of finite element models. Measured data such as natural

frequencies and mode shapes identified from vibration tests are used to evaluate the predicted data obtained from finite element analysis. There are usually discrepancies between the measured data and the predicted data. In model updating, the inaccurate mass, stiffness and damping parameters in the analytical model structure are corrected in order to match the analytical modal data with measured data.

2.10.1 Correction of System Matrices using Measured Modal Matrix

The measured modes for a given structure are usually non-orthogonal relative to the analytical mass matrix. These modes can be corrected to satisfy the orthogonality requirements. Several investigators have proposed methods for orthogonalisation of measured mode shapes. These methods have been developed to modify the measured modes which are used to correct the finite element models [91].

Baruch and Bar-Itzhack [91] proposed a method to force the measured mode shapes for symmetric systems to satisfy the analytical requirement of weighted orthonormality. The modified measured modal matrix Φ minimises the weighted Euclidean norm of the errors subject to the orthogonality requirement. The analytical mass matrix $\mathbf{M}(n \times n)$ is assumed to be symmetric and positive definite and is expected to be more accurate than the stiffness matrix. The weighted orthogonality condition can be written as

$$\Phi^T \mathbf{M} \Phi = \mathbf{I} \quad (2.164)$$

The method utilises a three stage process. In the first stage, the measured modes are normalised individually with respect to the analytical mass matrix. The measured mode vector ϕ_{mi} can be normalised using the following equation

$$\phi_{mi} = \tilde{\phi}_{mi} \left(\tilde{\phi}_{mi}^T \mathbf{M} \tilde{\phi}_{mi} \right)^{-1/2} \quad (2.165)$$

where $\tilde{\phi}_{mi}$ is the i^{th} measured mode vector before the normalisation. The measured modal matrix $\Phi_m = [\phi_{m1} \ \phi_{m2} \ \dots \ \phi_{mm}]$ consists of the m normalised measured modes.

In the second stage, the normalised measured modal matrix Φ_m is orthonormalised. The modified modes Φ are closest to the measured modes in a weighted Euclidean norm sense and satisfy equation (2.164). The optimal orthonormality of the measured mode shapes [91] can be written as

$$\Phi = \Phi_m \left(\Phi_m^T \mathbf{M} \Phi_m \right)^{-1/2} \quad (2.166)$$

In the third stage, the modified measured modes are used to update the computed stiffness matrix \mathbf{K} . The corrected stiffness matrix \mathbf{K}_{up} minimises the weighted Euclidean norm and satisfies the equilibrium dynamic condition

$$\mathbf{K}_{up} \Phi = \mathbf{M} \Phi \Lambda^2 \quad (2.167)$$

where $\mathbf{K}_{up} = \mathbf{K}_{up}^T$ is a corrected stiffness matrix which is a symmetric matrix and Λ is the measured frequencies. The corrected stiffness matrix can be expressed as

$$\mathbf{K}_{up} = \mathbf{K} - \left(\mathbf{M} \Phi \Phi^T \mathbf{K} + \mathbf{K} \Phi \Phi^T \mathbf{M} \right) + \mathbf{M} \Phi \left(\Phi^T \mathbf{K} \Phi + \Lambda^2 \right) \Phi^T \mathbf{M} \quad (2.168)$$

Berman [92] made suggestions to improve the method presented by Baruch and Bar-Izhack [91]. In this method, the reduced stiffness matrix by Guyan/Irons [52, 53] reduction is assumed more correct than the reduced mass matrix. The suggestion is to

use the analytical stiffness matrix to orthogonalise the measured modes. The modified measured modes are used to correct the analytical mass matrix.

The mass matrix is not the only possible choice as a reference base. Different possible methods are proposed [92, 93] to obey the orthogonal requirement. Two different methods are proposed by Baruch [94] for identification of dynamic structures. The stiffness matrix is considered to be a positive definite matrix. The measured modal matrix is selected as a reference base. The measured modal matrix is slightly modified based on the theoretical rigid body modal matrix Φ_R . The modified modes minimise the weighted Euclidean norm of $\mathbf{M}^{1/2}(\tilde{\Phi} - \Phi_m)$ and satisfy the constraint $\Phi_R^T \mathbf{M} \tilde{\Phi} = 0$. The slight modification of the flexible mode shapes [94] is expressed in the form

$$\tilde{\Phi} = [I - \Phi_R \Phi_R^T \mathbf{M}] \Phi_m \quad (2.169)$$

The first method considers the mass matrix which is more accurate than the stiffness matrix. The mass matrix is corrected to comply with the orthogonality condition and the stiffness matrix is corrected to satisfy the dynamic equations. The normalised flexible mode shape ϕ_i with respect to the analytical mass matrix is

$$\phi_i = \tilde{\Phi}_i \left(\tilde{\Phi}_i^T \mathbf{M} \tilde{\Phi}_i \right)^{-1/2} \quad (2.170)$$

The modified mode shapes $\Phi = [\phi_1 \ \phi_2 \ \dots \ \phi_m]$ are used to correct the analytical mass and stiffness matrices. The corrected mass matrix \mathbf{M}_{up} that is closest to the analytical mass matrix \mathbf{M} in a weighted Euclidean norm sense with the constraint of orthogonality from equation (2.164) can be expressed as

$$\mathbf{M}_{up} = \mathbf{M} - \mathbf{K}_{up} \Phi (\Phi^T \mathbf{K}_{up} \Phi)^{-1} (\Phi^T \mathbf{M} \Phi - \mathbf{I}) (\Phi^T \mathbf{K}_{up} \Phi)^{-1} \Phi^T \mathbf{K}_{up} \quad (2.171)$$

where the stiffness matrix \mathbf{K}_{up} is unknown. Substituting equilibrium equation (2.167) into equation (2.171) yields

$$\mathbf{M}_{up} = \mathbf{M} - \mathbf{M} \Phi (\Phi^T \mathbf{M} \Phi)^{-1} (\Phi^T \mathbf{M} \Phi - \mathbf{I}) (\Phi^T \mathbf{M} \Phi)^{-1} \Phi^T \mathbf{M} \quad (2.172)$$

The corrected stiffness matrix can be written as

$$\mathbf{K}_{up} = \mathbf{K} - \mathbf{K} \Phi \Phi^T \mathbf{M}_{up} - \mathbf{M}_{up} \Phi \Phi^T \mathbf{K} + \mathbf{M}_{up} \Phi \Phi^T \mathbf{K} \Phi \Phi^T \mathbf{M}_{up} + \mathbf{M}_{up} \Phi \Lambda^2 \Phi^T \mathbf{M}_{up} \quad (2.173)$$

The second method considers the stiffness matrix to be more accurate than the mass matrix. The stiffness matrix is corrected to comply with the orthogonality requirement and the mass matrix is corrected to satisfy the dynamic equations. The normalised flexible mode shape ϕ_i with respect to the analytical stiffness matrix is

$$\phi_i = \omega_i \tilde{\phi}_i (\tilde{\phi}_i^T \mathbf{K} \tilde{\phi}_i)^{-1/2} \quad (2.174)$$

where ω_i is the i^{th} measured frequency. The corrected stiffness matrix \mathbf{K}_{up} is closest to the analytical stiffness matrix in natural weighted norm sense with constraint $\Phi^T \mathbf{K}_{up} \Phi = \Lambda^2$.

$$\mathbf{K}_{up} = \mathbf{K} - \mathbf{M}_{up} \Phi (\Phi^T \mathbf{K} \Phi - \Lambda^2) \Phi^T \mathbf{M}_{up} \quad (2.175)$$

where the mass matrix \mathbf{M}_{up} is unknown. Substituting equilibrium equation (2.167) into equation (2.175) yields

$$\mathbf{K}_{up} = \mathbf{K} - \mathbf{K} \Phi (\Phi^T \mathbf{K} \Phi)^{-1} (\Phi^T \mathbf{K} \Phi - \Lambda^2) (\Phi^T \mathbf{K} \Phi)^{-1} \Phi^T \mathbf{K} \quad (2.176)$$

The corrected mass matrix can be expressed as

$$\mathbf{M}_{up} = \begin{pmatrix} \mathbf{M} - \mathbf{M} \Phi \Lambda^{-2} \Phi^T \mathbf{K}_{up} - \mathbf{K}_{up} \Phi \Lambda^{-2} \Phi^T \mathbf{M} + \\ \mathbf{K}_{up} \Phi \Lambda^{-4} \Phi^T \mathbf{K} + \mathbf{K}_{up} \Phi \Lambda^{-2} \Phi^T \mathbf{M} \Phi \Lambda^{-2} \Phi^T \mathbf{K} \end{pmatrix} \quad (2.177)$$

More details are presented in references [91, 94].

Optimisation methods of mass and stiffness matrices using modal data have been developed. The improved matrices satisfy the orthogonality requirements and equation of motion and reproduce exactly the measured modes. However, the corrected matrices are to be close to the analytical model in a weighted Euclidean norm sense and with certain assumptions on the Lagrange multipliers. O'Callahan and Leung [95] proposed a method based on using the generalised inverse (*the pseudo inverse*) of the measured modal matrix.

The modified mass matrix \mathbf{M}_{up} which satisfies the orthogonality condition can be written as

$$\mathbf{M}_{up} = \mathbf{M} + \mathbf{M}^{1/2} \boldsymbol{\Phi}^+ (\mathbf{I} - \hat{\mathbf{M}}) \boldsymbol{\Phi}^{+T} \mathbf{M}^{1/2} \quad (2.178)$$

where $\hat{\mathbf{M}} = \boldsymbol{\Phi}_m^T \mathbf{M} \boldsymbol{\Phi}_m$ and $\boldsymbol{\Phi}^+$ is the generalised measured normal modes $\boldsymbol{\Phi} = \boldsymbol{\Phi}_m^T \mathbf{M}^{1/2}$.

$$\boldsymbol{\Phi}^+ = \mathbf{M}^{1/2} \boldsymbol{\Phi}_m \hat{\mathbf{M}}^{-1} \quad (2.179)$$

The modified stiffness matrix \mathbf{K}_{up} which satisfies equation (2.167) can be expressed as

$$\mathbf{K}_{up} = \mathbf{K} - (\mathbf{M}_{up} \boldsymbol{\Phi}_m \boldsymbol{\Phi}_m^T \mathbf{K} + \mathbf{K} \boldsymbol{\Phi}_m \boldsymbol{\Phi}_m^T \mathbf{M}_{up}) + \mathbf{M}_{up} \boldsymbol{\Phi}_m (\boldsymbol{\Phi}_m^T \mathbf{K} \boldsymbol{\Phi}_m + \Lambda^2) \boldsymbol{\Phi}_m^T \mathbf{M}_{up} \quad (2.180)$$

2.10.2 Computational Model Updating

In large structures, measurements at a large number of locations are costly and require a large amount of data to be processed [90]. Assessing the effects of parameter changes on the finite element data can be complex if the number of parameters becomes too large. Computational model updating procedures are used

based on adjusting several parameters to minimise the discrepancies between the measured and predicted data.

For an undamped system, changes of the analytical model matrices [18] can be represented as

$$\mathbf{M}(\boldsymbol{\theta}) = \mathbf{M}_o + \theta_1 \mathbf{M}_1 + \theta_2 \mathbf{M}_2 + \dots + \theta_p \mathbf{M}_p = \mathbf{M}_o + \sum_{j=1}^p \theta_j \mathbf{M}_j \quad (2.181)$$

$$\mathbf{K}(\boldsymbol{\theta}) = \mathbf{K}_o + \theta_1 \mathbf{K}_1 + \theta_2 \mathbf{K}_2 + \dots + \theta_p \mathbf{K}_p = \mathbf{K}_o + \sum_{j=1}^p \theta_j \mathbf{K}_j \quad (2.182)$$

where $\{\mathbf{K}_o, \mathbf{M}_o\}$ define the original mass and stiffness matrices. $\{\mathbf{K}_j, \mathbf{M}_j\}$ substructure matrices define the location and error of model uncertainties, $\boldsymbol{\theta} = (\theta_1; \theta_2; \dots; \theta_p)$ represents an unknown vector of uncertainty parameters and the subscript “ p ” denotes the number of unknown parameters in the vector $\boldsymbol{\theta}$ where $p < 2n^2$.

There are a number of different possibilities which might reduce the residuals to zero. The residuals can be written as

$$\mathbf{r}(\boldsymbol{\theta}) = O_m - O_a(\boldsymbol{\theta}) \quad (2.183)$$

O_m is a vector of measured data, $O_a(\boldsymbol{\theta})$ is the corresponding predicted analysis data vector. The residuals are nonlinear and depend on the uncertainty parameters

$$\mathbf{r}(\boldsymbol{\theta} + \Delta\boldsymbol{\theta}) = \mathbf{r}_o + \mathbf{S}\Delta\boldsymbol{\theta} \quad (2.184)$$

where \mathbf{r}_o is the predicted analysis data vector at $\boldsymbol{\theta}_o$, $\mathbf{S} = \partial\mathbf{r}/\partial\boldsymbol{\theta}|_{\boldsymbol{\theta}=\boldsymbol{\theta}_o}$ is the sensitivity matrix and $\Delta\boldsymbol{\theta} = (\boldsymbol{\theta} - \boldsymbol{\theta}_o)$ is the change of uncertainty parameters vector.

Schedlinski *et al.* [90] presented the residuals used in computational model updating which are based on the natural frequency and mode shape residuals. The residual vector can be written in the form

$$\mathbf{r}(\boldsymbol{\theta}) = \begin{bmatrix} (\boldsymbol{\Lambda}_m - \boldsymbol{\Lambda}) \\ (\boldsymbol{\Phi}_m - \boldsymbol{\Phi}) \end{bmatrix}_{\boldsymbol{\theta}} \quad (2.185)$$

where $\boldsymbol{\Lambda}_m$ and $\boldsymbol{\Lambda}(\boldsymbol{\theta})$ are the measured and predicted vectors of natural frequencies, $\boldsymbol{\Phi}_m$ and $\boldsymbol{\Phi}(\boldsymbol{\theta})$ are the measured and predicted mode shape vectors. The errors of the residuals vector can be simply calculated in the general case by minimising the objective function

$$E = \mathbf{r}^T \mathbf{W} \mathbf{r} + \boldsymbol{\theta}^T \mathbf{W}_\theta \boldsymbol{\theta} \quad (2.186)$$

where $\{\mathbf{W}, \mathbf{W}_\theta\}$ are weighting matrices. For $\mathbf{W}_\theta = 0$, equation (2.186) represents a standard weighted least squares of the residuals vector.

$$E = \mathbf{r}^T \mathbf{W} \mathbf{r} \quad (2.187)$$

The sensitivity matrix [96] is given as

$$\mathbf{S} = \begin{bmatrix} \partial \boldsymbol{\Lambda} / \partial \boldsymbol{\theta} \\ \partial \boldsymbol{\Phi} / \partial \boldsymbol{\theta} \end{bmatrix}_{\boldsymbol{\theta} = \boldsymbol{\theta}_0} \quad (2.188)$$

The sensitivity matrix \mathbf{S} contains the first derivative of eigenvalues and eigenvectors with respect to the parameters $\boldsymbol{\theta}$. The updated parameters $\partial \boldsymbol{\theta}$ can be calculated as

$$\partial \boldsymbol{\theta} = -\mathbf{S}^+ \partial \mathbf{r} \quad (2.189)$$

where \mathbf{S}^+ is the pseudo inverse of the sensitivity matrix \mathbf{S} .

2.10.3 Calculating the Sensitivity Matrix Analytically

Calculating the sensitivity matrix analytically is of interest to most investigators. There is already substantial literature on eigenvalue and eigenvector derivatives for undamped systems. Fox and Kapoor [97] provided a method applicable to symmetric undamped systems. The derivative of the eigenvector equation with respect to the j^{th} parameter θ_j can be expressed as

$$\left(\frac{\partial \mathbf{K}}{\partial \theta_j} - \lambda_i \frac{\partial \mathbf{M}}{\partial \theta_j} \right) \phi_i - \frac{\partial \lambda_i}{\partial \theta_j} \mathbf{M} \phi_i + (\mathbf{K} - \lambda_i \mathbf{M}) \frac{\partial \phi_i}{\partial \theta_j} = 0 \quad (2.190)$$

The derivative of the i^{th} eigenvalue λ_i can be calculated by pre-multiplying equation (2.190) by the transpose of the eigenvector ϕ_i^T and using the mass orthogonality

$$\frac{\partial \lambda_i}{\partial \theta_j} = \phi_i^T \left(\frac{\partial \mathbf{K}}{\partial \theta_j} - \lambda_i \frac{\partial \mathbf{M}}{\partial \theta_j} \right) \phi_i \quad (2.191)$$

The derivatives of eigenvectors are expressed by Fox and Kapoor [97] as a linear combination of all eigenvectors.

$$\dot{\Phi} = \frac{\partial \Phi}{\partial \theta} = \Phi C \quad (2.192)$$

The derivative of the i^{th} eigenvector ϕ_i with respect to the j^{th} parameter θ_j can be calculated using equation (2.190) and the derivative of the mass orthogonality equation (2.164). These expressions have been expanded by numerous authors, e.g. [98-103] to determine eigenvalue and eigenvector derivatives for more general non-symmetric undamped systems.

Nelson [104] simplified the procedure for calculating eigenvector derivatives of undamped systems so that only the eigenvalue and eigenvector under consideration

are required. The implementation and application of sensitivity approaches are discussed by Allen and Martinez [105]. The sensitivity techniques are used to improve aircraft models using ground vibration test data [106].

2.10.4 Ball of convergence for model updating

Updating based on eigenvalues and eigenvectors may converge if values of $\{\boldsymbol{\theta}_j\}$ in equations (2.181) and (2.182) are very small and if the $\{\mathbf{K}_j, \mathbf{M}_j\}$ matrices are small relative to the original system matrices $\{\mathbf{K}_o, \mathbf{M}_o\}$. It might be possible to get small values of errors (sum of the squares of the residuals) and to update the analytical eigenvalues and eigenvectors to match the measured eigenvalues and eigenvectors. However, if the analytical system matrices are far from the original system the updating based on eigenvalues and eigenvectors may not converge.

If there are further values of convergence, these values define the convergence interval. If this interval is infinite, it forms a radius of convergence [40]. The minimised from far values of the errors (residuals) and the converged model updating form a ball of convergence around the original point.

2.11 Conclusion

This chapter has reviewed the areas of interest for to this project and highlighted some of the conventional techniques. This work attempts to develop and generalise model updating methods based on cross orthogonality for generally damped systems.

This project also addresses eigenvalue and eigenvector derivative methods. The developed methods have been introduced to overcome the difficulties facing conventional techniques, and the following chapters will address these methods in detail.

CHAPTER 3. 3-D Finite Element Modelling to develop dynamic models

The finite element (F.E.) method is well-known in the analysis of continuum and structural mechanics [107]. It is widely used in the aerospace industry and it has become an accepted analysis method for solving eigenvalue problems in order to compute the natural frequencies and the mode shapes of the system [18]. In recent years, finite element model information has been combined with physical test model information for model updating.

The F.E. method begins with dividing the structure into a large number of small elements. The equations and relationships are derived that exactly describe the behaviour of each small part of the structure. Generally, the numerical solution becomes more accurate as element sizes are reduced [107, 108].

The ambitions of this chapter are to identify the structural models and dynamic behaviour of “GARTEUR-like” aircraft structure. This chapter presents a general overview of finite element modelling and focuses on 3-D 20-noded quadratic hexahedron (brick) element. The structural analysis has been carried out on the “GARTEUR-like” structure (shown in Figure 3-1) at the University of Nottingham laboratory. GARTEUR is an abbreviation for a group for aeronautical research and technology in Europe.

The structure is made from Aluminium. The length of the structure is 1.5 m with thickness 50mm, and the wing span is 2.0 m with thickness 10mm (more details

about the geometry is in ref. [20]). The structure was manufactured by staff in the university workshops. The overall mass is 42.20 kg.

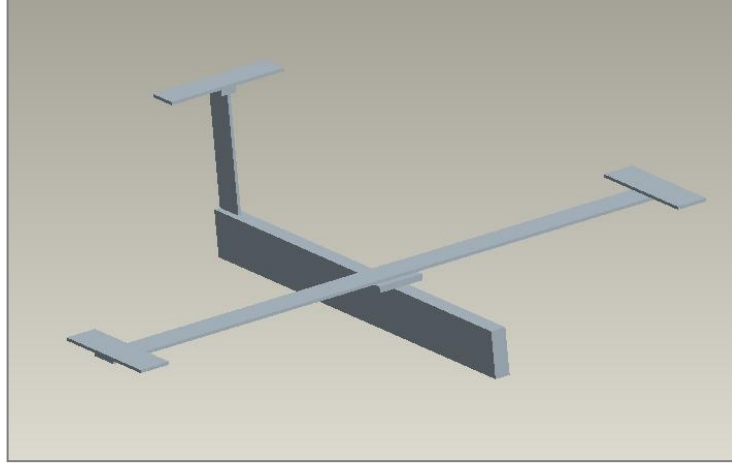


Figure 3-1 : GARTEUR-like aircraft structure

3.1 Derivation of finite element formulation

The derivation of the finite element formulation of structural problems can be broken into a number of steps [107].

Step 1: Define the element and its shape functions. The first step in any finite element analysis is to divide the structure into elements and examine the behaviour of a typical element.

Step 2: Satisfy the material law. The element stiffness can be derived from the differentiation of the displacements (to obtain the strains) and then use the material law to calculate the stress. The generalized Hooke's law can be used to express the stress-strain relationship.

$$\boldsymbol{\sigma} = \mathfrak{M}_{pro} \boldsymbol{\varepsilon} \quad (3.1)$$

$$\boldsymbol{\varepsilon} = \mathbf{B} \mathbf{q}_e \quad (3.2)$$

where \mathfrak{M}_{pro} is material property matrix. $\boldsymbol{\varepsilon}$ is the element strain vector, \mathbf{B} is strain shape function matrix.

Step 3: Derive the element stiffness matrix. An energy approach is used to derive the element stiffness matrix. Using the total potential energy \mathbf{P}_{eng} , the differential of \mathbf{P}_{eng} with respect to the displacement \mathbf{q}_e must be zero.

$$\mathbf{P}_{eng} = \int_v \frac{1}{2} \mathbf{q}_e^T \mathbf{B}^T \mathfrak{M}_{pro} \mathbf{B} \mathbf{q}_e dv - \mathbf{q}_e^T \mathbf{f}_e \quad (3.3)$$

The differential of \mathbf{P}_{eng} in equation (3.3) with respect to the displacement \mathbf{q}_e must be zero.

$$\frac{\partial \mathbf{P}_{eng}}{\partial \mathbf{q}_e} = 0 \quad (3.4)$$

Then, the element stiffness matrix can be calculated from equation (3.4)

$$\mathbf{K}_e \mathbf{q}_e = \mathbf{f}_e \quad , \quad \mathbf{K}_e = \int_v \mathbf{B}^T \mathfrak{M}_{pro} \mathbf{B} dv \quad (3.5)$$

Step 4: Assemble the overall stiffness matrix. The individual elements are assembled together to form the overall structure. The displacement of a particular node must be the same for every element connected to it. The externally applied forces at the nodes must be balanced by the forces on the elements at these nodes.

$$\mathbf{f}_{external} = \sum_{elements} \mathbf{f}_e = \sum_{elements} \mathbf{K}_e \mathbf{q}_e \quad (3.6)$$

The global system of equations can be written as

$$\mathbf{K}_{global} \mathbf{u}_{global} = \mathbf{f}_{global} \quad (3.7)$$

Step 5: Apply the boundary conditions and external loads. To obtain a unique solution of the problem, some constraints and loading conditions must be applied at some of the nodes. The boundary conditions can be included in the system of linear algebraic equations which can be solved to obtain a unique solution for the displacements at each node.

Step 6: Solve the equations. The algebraic equations can be solved by any standard solver such as Gaussian elimination technique. A unique solution is obtained because the number of unknown variables (displacements) at each node is equal the number of equations.

Step 7: Compute other variables. After solving the global equations, displacements at all the nodal points are determined. Solving the eigenvalue problem gives the eigenvalues and eigenvectors of the system.

3.2 Structural problems analysis

The solution of the stress and strain distributions is required in the FE formulation for continuum elements. The degrees of freedom depend on the displacement components which are independent variables and an approximation for the displacement function within each element is assumed [107]. In many applications, the problems are solved using two-dimensional plane stress or strain distributions, axisymmetric solid, plate bending and shells, or full three-dimensional solid models.

3.2.1 Two-dimensional problems

It is sometimes possible to approximate a 3-D problem to a simpler 2-D application in which only the x - y plane is modelled. All 2-D solutions are approximations of 3-D solutions. The simplest 2-D finite element analysis is three nodes which is called triangle element. The four nodes element is called quadrilateral elements. [109]. Two assumptions about the stress and thickness in the z -direction are made as follows:

Plane stress is used to define thin geometries in the z -direction where $\sigma_{zz} = 0$. Plane strain is used to define very thick geometries in the z -direction where $\epsilon_{zz} = 0$ but $\sigma_{zz} \neq 0$. The simplest two dimensional continuum elements are 3-node triangles with straight line sides. Alternatively, 4-node straight sided a quadrilateral element can be used [107].

3.2.2 Three- dimensional problems

Three dimensional continuum elements can be used to model all practical problems since no dimensional approximation is assumed. The simplest three dimensional element is a 4-node constant strain tetrahedron with a linear variation of displacement. 8-node hexahedron element and 20-node brick elements [108] are also used as shown in Figure 3-2.

3.3 FE modelling 3-D

In this chapter, we focus only on 3-D finite elements. The structure is divided into a large number of small elements. Each element has 20-node points and each node has

three translations degrees of freedom. Figure 3-2 show a 3-D 20-noded quadratic hexahedron element (brick). The front face $\{1,9,2,11,4,12,3,10\}$ is in + z direction, the back face $\{5,17,6,19,8,20,7,18\}$ is in - z direction and mid-side nodes $\{13,14,15,16\}$ are in middle of z direction.

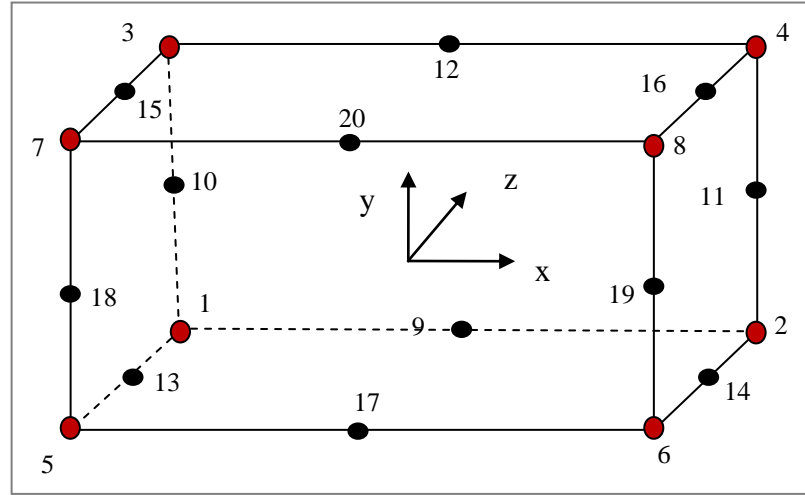


Figure 3-2 : Quadratic hexahedron (Brick) 20-Noded element

3.3.1 Shape functions

The shape functions are interpolation functions for the displacement function that must satisfy three conditions. The shape function N_i has to be 1 at the node i itself ($N_i = 1$ at node i). The shape function N_i has to be 0 at nodes other than i node ($N_i = 0$ at node points $j=1:20$ and $j \neq i$ where 20 is the number of node points in the element). The summation of all shape functions should be 1 for all (η, ξ, ζ) [108].

Figure 3-3 shows reference coordinates (η, ξ, ζ) for the 20-noded 3-D brick element.

The shape function of the corner node points are shown in Figure 3-3 and formulated on the following equations [108].

$$\left. \begin{aligned} N_1 &= -(1-\eta)(1+\zeta)(1-\xi)(2+\eta-\zeta+\xi)/8.0 \\ N_2 &= -(1+\eta)(1+\zeta)(1-\xi)(2-\eta-\zeta+\xi)/8.0 \\ N_3 &= -(1-\eta)(1+\zeta)(1+\xi)(2+\eta-\zeta-\xi)/8.0 \\ N_4 &= -(1+\eta)(1+\zeta)(1+\xi)(2-\eta-\zeta-\xi)/8.0 \end{aligned} \right\} \quad (3.8)$$

$$\left. \begin{aligned} N_5 &= -(1-\eta)(1-\zeta)(1-\xi)(2+\eta+\zeta+\xi)/8.0 \\ N_6 &= -(1+\eta)(1-\zeta)(1-\xi)(2-\eta+\zeta+\xi)/8.0 \\ N_7 &= -(1-\eta)(1-\zeta)(1+\xi)(2+\eta+\zeta-\xi)/8.0 \\ N_8 &= -(1+\eta)(1-\zeta)(1+\xi)(2-\eta+\zeta-\xi)/8.0 \end{aligned} \right\} \quad (3.9)$$

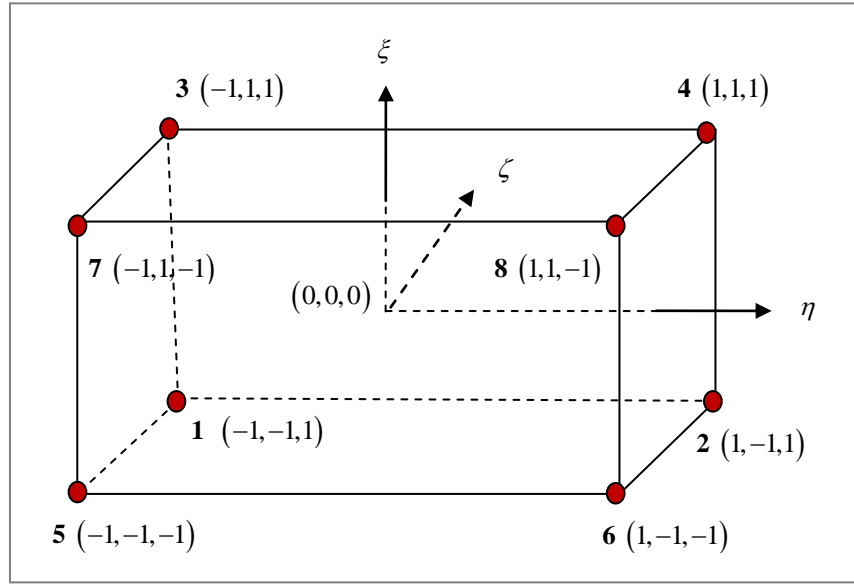


Figure 3-3 : Reference coordinates for the corner node points of 20-noded element

Figure 3-4 shows the reference coordinates (η, ξ, ζ) for the middle points. The shape functions of the middle node points are formulated in the following equations [108]

$$\left. \begin{aligned} N_9 &= (1-\eta^2)(1-\xi)(1+\zeta)/4.0 \\ N_{10} &= (1-\xi^2)(1-\eta)(1+\zeta)/4.0 \\ N_{11} &= (1-\xi^2)(1+\eta)(1+\zeta)/4.0 \\ N_{12} &= (1-\eta^2)(1+\xi)(1+\zeta)/4.0 \end{aligned} \right\} \quad (3.10)$$

$$\left. \begin{aligned} N_{13} &= (1-\zeta^2)(1-\eta)(1-\xi)/4.0 \\ N_{14} &= (1-\zeta^2)(1+\eta)(1-\xi)/4.0 \\ N_{15} &= (1-\zeta^2)(1-\eta)(1+\xi)/4.0 \\ N_{16} &= (1-\zeta^2)(1+\eta)(1+\xi)/4.0 \end{aligned} \right\} \quad (3.11)$$

$$\left. \begin{aligned} N_{17} &= (1-\eta^2)(1-\xi)(1-\zeta)/4.0 \\ N_{18} &= (1-\xi^2)(1-\eta)(1-\zeta)/4.0 \\ N_{19} &= (1-\xi^2)(1+\eta)(1-\zeta)/4.0 \\ N_{20} &= (1-\eta^2)(1+\xi)(1-\zeta)/4.0 \end{aligned} \right\} \quad (3.12)$$

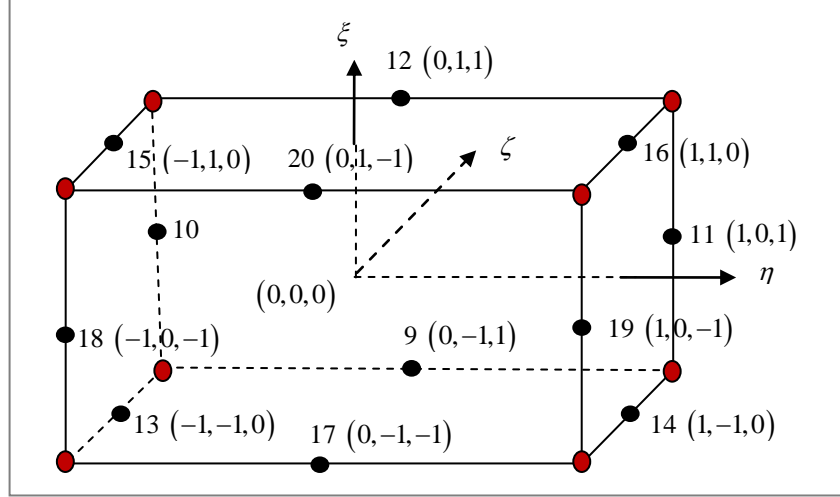


Figure 3-4 : Reference coordinates for middle node points of 20-noded element

3.3.2 Natural coordinate system

The interpolation functions in equations (3.8)-(3.12) are functions of $\{\eta, \xi, \zeta\}$. The $\{x, y, z\}$ coordinates can be described as functions of $\{\eta, \xi, \zeta\}$ using their nodal point values as

$$\begin{aligned} x(\eta, \xi, \zeta) &= \sum_{i=1}^{20} N_i(\eta, \xi, \zeta) x_i \\ y(\eta, \xi, \zeta) &= \sum_{i=1}^{20} N_i(\eta, \xi, \zeta) y_i \\ z(\eta, \xi, \zeta) &= \sum_{i=1}^{20} N_i(\eta, \xi, \zeta) z_i \end{aligned} \quad (3.13)$$

where $\{x_i, y_i, z_i\}$ are nodal displacements in the $\{x, y, z\}$ directions. The derivative of the $\{x, y, z\}$ coordinates in equations (3.13) with respect to $\{\eta, \xi, \zeta\}$ can be written as

$$\begin{aligned}
\frac{\partial x}{\partial \eta} &= \sum_{i=1}^{20} \frac{\partial N_i}{\partial \eta} x_i, & \frac{\partial x}{\partial \xi} &= \sum_{i=1}^{20} \frac{\partial N_i}{\partial \xi} y_i, & \frac{\partial x}{\partial \zeta} &= \sum_{i=1}^{20} \frac{\partial N_i}{\partial \zeta} z_i \\
\frac{\partial y}{\partial \eta} &= \sum_{i=1}^{20} \frac{\partial N_i}{\partial \eta} x_i, & \frac{\partial y}{\partial \xi} &= \sum_{i=1}^{20} \frac{\partial N_i}{\partial \xi} y_i, & \frac{\partial y}{\partial \zeta} &= \sum_{i=1}^{20} \frac{\partial N_i}{\partial \zeta} z_i \\
\frac{\partial z}{\partial \eta} &= \sum_{i=1}^{20} \frac{\partial N_i}{\partial \eta} x_i, & \frac{\partial z}{\partial \xi} &= \sum_{i=1}^{20} \frac{\partial N_i}{\partial \xi} y_i, & \frac{\partial z}{\partial \zeta} &= \sum_{i=1}^{20} \frac{\partial N_i}{\partial \zeta} z_i
\end{aligned} \tag{3.14}$$

The displacement function can be expressed as follows using the same shape functions

$$\begin{aligned}
\mathbf{q}_x(\eta, \xi, \zeta) &= \sum_{i=1}^{20} N_i(\eta, \xi, \zeta) \mathbf{q}_{xi} \\
\mathbf{q}_y(\eta, \xi, \zeta) &= \sum_{i=1}^{20} N_i(\eta, \xi, \zeta) \mathbf{q}_{yi} \\
\mathbf{q}_z(\eta, \xi, \zeta) &= \sum_{i=1}^{20} N_i(\eta, \xi, \zeta) \mathbf{q}_{zi}
\end{aligned} \tag{3.15}$$

The first derivative of the shape function [108, 110] with respect to $\{\eta, \xi, \zeta\}$.

$$\begin{aligned}
\frac{\partial N_i}{\partial \eta} &= \frac{\partial N_i}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial \eta} + \frac{\partial N_i}{\partial z} \frac{\partial z}{\partial \eta} \\
\frac{\partial N_i}{\partial \xi} &= \frac{\partial N_i}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial \xi} + \frac{\partial N_i}{\partial z} \frac{\partial z}{\partial \xi} \\
\frac{\partial N_i}{\partial \zeta} &= \frac{\partial N_i}{\partial x} \frac{\partial x}{\partial \zeta} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial \zeta} + \frac{\partial N_i}{\partial z} \frac{\partial z}{\partial \zeta}
\end{aligned} \tag{3.16}$$

Equation (3.16) can be written using the Jacobian matrix [108]

$$\begin{bmatrix} \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{bmatrix} = \mathbf{J}_{ac} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{bmatrix} \tag{3.17}$$

To find the Cartesian derivatives of N_i , invert equation (3.17)

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{bmatrix} = \mathbf{J}_{ac}^{-1} \begin{bmatrix} \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \zeta} \end{bmatrix} \tag{3.18}$$

The Jacobian matrix is described as

$$\mathbf{J}_{ac} = \begin{bmatrix} \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} \sum \frac{\partial N_i}{\partial \eta} x_i & \sum \frac{\partial N_i}{\partial \eta} y_i & \sum \frac{\partial N_i}{\partial \eta} z_i \\ \sum \frac{\partial N_i}{\partial \xi} x_i & \sum \frac{\partial N_i}{\partial \xi} y_i & \sum \frac{\partial N_i}{\partial \xi} z_i \\ \sum \frac{\partial N_i}{\partial \zeta} x_i & \sum \frac{\partial N_i}{\partial \zeta} y_i & \sum \frac{\partial N_i}{\partial \zeta} z_i \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \dots \\ \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \dots \\ \frac{\partial N_1}{\partial \zeta} & \frac{\partial N_2}{\partial \zeta} & \dots \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \dots & \dots & \dots \end{bmatrix} \quad (3.19)$$

3.3.3 The stress-strain relationship

The element stiffness [110] can be derived from the differentiation of the displacements $\{q_x, q_y, q_z\}$ equation (3.15) to obtain the strains

$$\boldsymbol{\varepsilon} = \mathbf{B} \mathbf{q}_e \quad (3.20)$$

The strain shape function [111] matrix $\mathbf{B} = [\mathbf{B}_1 \quad \mathbf{B}_2 \quad \dots \quad \mathbf{B}_i]$

$$\mathbf{B} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & 0 & \dots & \frac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \dots & 0 & \frac{\partial N_i}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_1}{\partial z} & \dots & 0 & 0 & \frac{\partial N_i}{\partial z} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & 0 & \dots & \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial y} & \dots & 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_1}{\partial x} & \dots & \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial x} \end{bmatrix} \quad (3.21)$$

where i is the number of element node points. The element displacement can be written as

$$\mathbf{q}_e = [N_1 \quad N_1 \quad N_1 \quad \dots \quad N_i \quad N_i \quad N_i]^T \quad (3.22)$$

where N_i is the shape function which is in 3-D coordinates (x, y, z) , and $i = 1:n_p$

where n_p is number of element node points, in this work $n_p = 20$. Each element is divided into 60 Dofs.

The generalized Hooke's law for the stress- strain relationship [108] can be expressed as a function of displacements written in a more concise way as

$$\boldsymbol{\sigma} = \mathfrak{M}_{pro} \boldsymbol{\varepsilon} = \mathfrak{M}_{pro} \mathbf{B} \mathbf{q}_e \quad (3.23)$$

where $\boldsymbol{\sigma}$ is the stress vector which contains all stress components, $\boldsymbol{\varepsilon}$ is the strain vector. \mathfrak{M}_{pro} is called the material “*or elastic*” properties matrix. The stress-strain relationships are given by

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix} = \mathfrak{M}_{pro} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} \quad (3.24)$$

where $\{\sigma_x, \sigma_y, \sigma_z\}$ and $\{\varepsilon_x, \varepsilon_y, \varepsilon_z\}$ are normal stresses and strains respectively, and $\{\tau_{xy}, \tau_{yz}, \tau_{xz}\}$ and $\{\gamma_{xy}, \gamma_{yz}, \gamma_{xz}\}$ are shear stresses and strains respectively. Assuming that the material is isotropic [108, 111]

$$\mathfrak{M}_{pro} = \frac{2\mu}{(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \quad (3.25)$$

where ν is a Poisson's ratio (*no units*), E is Young's modulus (Nm^{-2}). The material properties $\{\nu, E\}$ are assumed to be constant in the element. μ is the shear modulus (Nm^{-2}).

$$\mu = \frac{E}{2(1+\nu)} \quad (3.26)$$

3.3.4 *Element stiffness matrix from potential energy*

In linear finite element analysis, the purpose of the structural analysis is to find a solution in which equilibrium is stable such that the system potential energy is a minimum. The potential energy is used in finite element analysis by expressing the problem in terms of the independent variables such as displacements [107]. The difference between the strain energy and the work done by the external forces on the structure is the potential energy which can be expressed as follows

$$\mathbf{P}_{eng} = \mathbf{S}_{eng} - \mathbf{W}_{done} \quad (3.27)$$

The strain energy \mathbf{S}_{eng} can be written as

$$\mathbf{S}_{eng} = \frac{1}{2} \mathbf{q}_e^T \mathbf{K}_e \mathbf{q}_e \quad (3.28)$$

The potential energy formulation is applied to the elements to derive the element stiffness matrix. The total potential energy (\mathbf{P}_{eng}) [108] of the element can be written as

$$\mathbf{P}_{eng} = \int_v \frac{1}{2} \sigma^T \varepsilon dv - \mathbf{q}_e^T \mathbf{f}_e \quad (3.29)$$

Substitute equations (3.23) and (3.20) into equation (3.29)

$$\mathbf{P}_{eng} = \int_v \frac{1}{2} (\mathfrak{M}_{pro} \mathbf{B} \mathbf{q}_e)^T (\mathbf{B} \mathbf{q}_e) dv - \mathbf{q}_e^T \mathbf{f}_e = \int_v \frac{1}{2} \mathbf{q}_e^T \mathbf{B}^T \mathfrak{M}_{pro} \mathbf{B} \mathbf{q}_e dv - \mathbf{q}_e^T \mathbf{f}_e \quad (3.30)$$

The differential of \mathbf{P}_{eng} with respect to the displacement \mathbf{q}_e must be zero.

$$\frac{\partial \mathbf{P}_{eng}}{\partial \mathbf{q}} = 0 = \int_v \mathbf{B}^T \mathfrak{M}_{pro} \mathbf{B} \mathbf{q}_e dv - \mathbf{f}_e \quad (3.31)$$

$$\mathbf{K}_e = \int_v \mathbf{B}^T \mathfrak{M}_{pro} \mathbf{B} dv \quad (3.32)$$

The stiffness and mass element matrices have dimension (60×60).

$$\mathbf{K}_e = \int_v \mathbf{B}^T \mathfrak{M}_{pro} \mathbf{B} dv = \int_x \int_y \int_z \mathbf{B}^T \mathfrak{M}_{pro} \mathbf{B} dx dy dz \quad (3.33)$$

To complete the evaluation of the integral, the element volume $dx dy dz$ in equation (3.33) is expressed in terms of reference coordinates $d\eta d\xi d\zeta$

$$dx dy dz = \det(\mathbf{J}_{ac}) d\eta d\xi d\zeta \quad (3.34)$$

Substitute equation (3.34) into equation (3.33) yields

$$\mathbf{K}_e = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \mathbf{B}^T \mathfrak{M}_{pro} \mathbf{B} \det(\mathbf{J}) d\eta d\xi d\zeta \quad (3.35)$$

where $\det(\mathbf{J}_{ac})$ is the determinant of the Jacobian matrix.

The Legendre-Gauss method is a technique which is often used to compute the numerical integration [110]. Equation (3.35) is evaluated by the Gauss numerical integration formula. Table 3-1 shows Gauss integration point coordinates and weight coefficients of the Gaussian quadrature formula for integration with 4-Gaussian point on a 1D line between ± 1 [108, 110].

Gauss points coordinates (η_i, ξ_j, ζ_k)	Gauss weight factors (w_i, w_j, w_k)
-0.86213631	0.34785485
-0.33998104	0.65214515
0.33998104	0.65214515
0.86213631	0.34785485

Table 3-1 : Gauss points and coefficients

The number of integration points required to evaluate the stiffness matrix for a hexahedron is 64 integration points. The 3-D element stiffness matrix [108] is

$$\mathbf{K}_e = \sum_i^{GP} \sum_j^{GP} \sum_k^{GP} \det(\mathbf{J}_{ac(\eta_i, \xi_j, \zeta_k)}) w_i w_j w_k \mathbf{B}_{(\eta_i, \xi_j, \zeta_k)}^T \mathfrak{M}_{pro} \mathbf{B}_{(\eta_i, \xi_j, \zeta_k)} \quad (3.36)$$

The 3-D element mass matrix [108] is

$$\mathbf{M}_e = \rho \sum_i^{GP} \sum_j^{GP} \sum_k^{GP} \det(\mathbf{J}_{ac(\eta_i, \xi_j, \zeta_k)}) w_i w_j w_k \mathbf{q}_{e(\eta_i, \xi_j, \zeta_k)} \mathbf{q}_{e(\eta_i, \xi_j, \zeta_k)}^T \quad (3.37)$$

where w_i , w_j and w_k denote Gauss weight coefficients and η_i , ξ_j and ζ_k denote Gauss integration points. GP represents the number of Gauss points on each scalar. ρ is material density.

3.3.5 Element stiffness matrix Assembly

The individual elements stiffness and mass matrices are assembled together to form the overall structure matrices. The global stiffness matrix dimension comprises the total nodal points of the structure multiplied by the number of degrees of freedom for each node. For n nodes with three degrees of freedom per node, the global stiffness matrix \mathbf{K} is of size $(3n \times 3n)$, while the global displacement \mathbf{q} and \mathbf{f} vectors are of size $(3n \times 1)$.

$$\frac{\partial \mathbf{P}_{eng\ overall}}{\partial \mathbf{q}} = \sum_{elements} \left(\frac{\partial \mathbf{P}_{eng}}{\partial \mathbf{q}} \right) = 0 \quad (3.38)$$

The global system of equations can be written as

$$\mathbf{K}_{global} = \sum_{elements} \mathbf{K}_e, \quad \mathbf{M}_{global} = \sum_{elements} \mathbf{M}_e \quad (3.39)$$

3.4 Structural analysis dimensions and properties

The structural analysis has been carried out on the ‘‘Aircraft-like’’ structure. The model material used is Aluminium. The length of the airplane is 1.5 m, and the wing span is 2.0 m with thickness 10mm. In order to solve the eigenvalue problem using the finite element method, the airplane model is divided into a large number of small

parts (finite elements) in 3 dimensional coordinates. The total number of nodes is 33753 and the total number of elements is 3232.

3.4.1 Meshing the structure

Mesh generation of the continuum region depends upon the shape of the element and the number of nodes for each element. The accurate solution of the finite elements converge to the accuracy of the final solution [109]. A MATLAB script called FE_GEOM.m is created to generate the geometry (node points) and to build-up the 20-noded quadratic hexahedron (brick) elements for the airplane structure. Node points are generated along the z-direction, x-direction and then in the y-direction. Each node point is labelled and has three degrees of freedom (x, y, z) that are saved in the matrix called “NodeSet”.

The structure is divided into small finite elements (Figure 3-5). The mesh is created by dividing the model structure into a number of meshable regions. The elements are built-up in the x-direction, y-direction and then in the z-direction. Each element has 20 labels related to each node in the element that is saved in a matrix called EleSet.

The node points in each element are ordered. The front face of the element represents the node points (1,9,2,11,4,12,3,10) which is in the positive direction of the z axis. The back face represents node points (5,17,6,19,8,20,7,18) which is in negative direction of z. The middle-side faces represent node points (13,14,15,16) which are in $z = 0$.

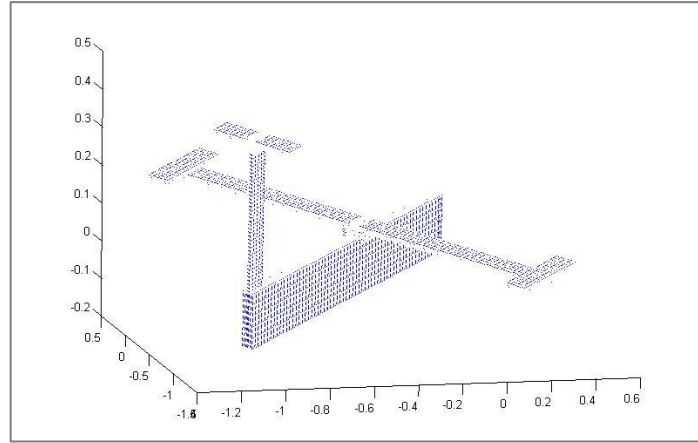


Figure 3-5 : Finite element meshing

3.4.2 Material Properties and an appropriate element ordering

The second MATLAB script called FE_PREP.m loads up the model data from the file called model_0.mat. In this script a few steps have been performed on each element and the associated node points. The first step here establishes the material properties of each element then, removes the redundant node points from the element topologies. Two columns are added onto the front of the EleSet matrix. The first column is to identify the element type (20-noded brick element) and the second column is to identify the material property code for each element. The node points NodeSet, number of nodes, element set EleSet, external faces Nface, number of faces and table of material properties MATTABL are saved into a file called model_1.mat.

3.4.2.1 Material Properties

The properties of the material [112] used for the model are shown in Table 3-2. These material properties are created in a matrix and related to the equation (3.25).

Material Properties	Units of Measure
Density ρ	2600-2800 kg/m ³
Elastic (Young's) Modulus =: E	70-79 GPascal (10 ⁹ Nm ⁻²)
Poisson's Ratio =: ν	0.33

Table 3-2 : Material properties of aluminium

3.4.2.2 *Removing redundant nodes from the element*

The generated elements have some doubled node points which have the same coordinates with different labels. In this section, there are two steps which are used to remove the redundant node points from the elements. The first step is to find all the doubled nodes in the matrix NodeSet which are closer to each other using a MATLAB function called reds_dbnodes.m and save the label of those nodes into a matrix called reds. The second step is to replace the label for the closer nodes in the element matrix EleSet by using a MATLAB function called kill_reds.m.

3.4.2.3 *Extracting external faces for the model structure*

The number of faces for each element is defined. Each element has 6 faces as in Figure 3-2 and each face has 8 node points as the following.

$$\begin{aligned}
 \text{face}(1) &= [1, 9, 2, 11, 4, 12, 3, 10] \quad , \quad \text{face}(4) = [3, 15, 7, 20, 8, 16, 4, 12] \\
 \text{face}(2) &= [5, 17, 6, 19, 8, 20, 7, 18] \quad , \quad \text{face}(5) = [1, 10, 3, 15, 7, 18, 5, 13] \\
 \text{face}(3) &= [1, 13, 5, 17, 6, 14, 2, 9] \quad , \quad \text{face}(6) = [2, 11, 4, 16, 8, 19, 6, 14]
 \end{aligned}$$

The repeated faces are removed from the total number of faces for all elements. A MATLAB function is used to identify which node points from the NodeSet matrix are not mentioned in the topologies of elements EleSet. The external faces with labels for each node are in a matrix. The number of external faces of the structure Nf is 3320. The node points which are not used in any element are removed from the

NodeSet matrix. The number of redundant node points removed is 15396 and the remaining node points are 18357.

3.4.2.4 *An appropriate element ordering*

In order to rearrange all the elements according to the selected Cartesian coordinates (x, y, z) , the global origin point of the structure is chosen $(-F_x/2, 0, 0)$, where F_x is the width of the fuselage (Figure 3-6). The priority element is chosen as the closest element to the original point by finding the square root of the sum of the squares of the difference between the mean value of the each element node and the original points. If the mean value for each element nodes is m_{xyz} and the difference between m_{xyz} and the original point is m_p , then the priority element is at the smallest value of the $norm(m_p)$. This is to get the minimum possible maximum front size.

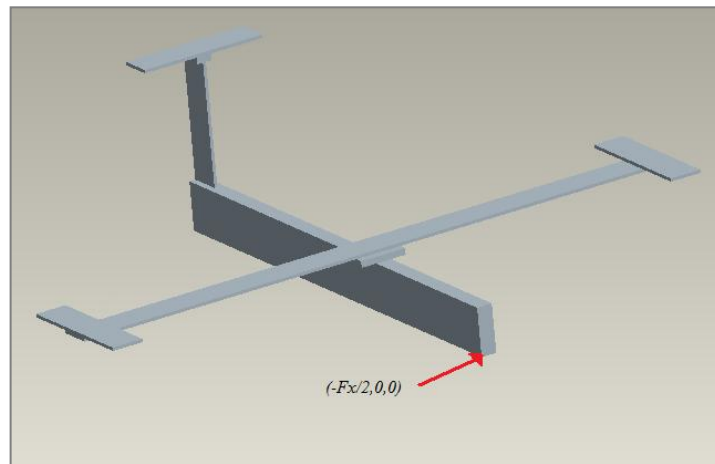


Figure 3-6 : Global origin point of the structure

3.4.3 *Derive the element matrices*

The third MATLAB script called FE_PROC loads up the model data from the file called model_1.m. In this script, there are a few steps which have been performed to calculate the dynamic model mass and stiffness for each element. The MATLAB function called sweep0 is used to identify the minimum number of degrees of freedom or the minimum front size for each element. The MATLAB function called sweep1 is used to assemble the reduced system matrices $\{\mathbf{K}, \mathbf{M}\}$ which represent stiffness and mass respectively. The system matrices are reduced using dynamic reduction during the element merge. The reduced stiffness and mass matrices have dimension (1473×1473).

3.4.3.1 *Element degrees of freedom*

Each element has 20 labels related to the corresponding node points. The number of degrees of freedom for each element is (20×3) degrees of freedom. The degrees of freedom of all elements are prepared in a matrix called Ele_DoFs related to each element degree of freedom index. The labels related to each $\{x, y, z\}$ degree of freedom for each element are stored in columns as follows $x = (1:3:58)$, $y = (2:3:59)$ and $z = (3:3:60)$. A MATLAB function is used to find the index for each element node points label from the matrix NodeSet.

$$\begin{aligned}x_{indx} &= indx \times 3 - 2 \\y_{indx} &= indx \times 3 - 1 \\z_{indx} &= indx \times 3\end{aligned}\tag{3.40}$$

3.4.4 *Maximum front size*

The MATLAB function called `sweep0` has two steps. The first step is to count the number of elements attached to each degree of freedom. The number of elements attached to each degree of freedom dictates how many elements are connected to each individual degree of freedom this number is kept in a vector called `Ele_count0`. The second step is to minimise the number of degrees of freedom that are in the global matrices. The elements are dealt with in an order which depends on the global original point. The labels of the degrees of freedom of each element that are present in the matrices during the merge and reduction are kept in a matrix called `DoFMap0`. As an element is merged into the matrices the number of remaining elements attached to each individual degree of freedom in that element is checked with the vector `Ele_count0`. The independent degrees of freedom are released from `DoFMap0`. The degrees of freedom of the next element are inserted in the available place in `DoFMap`. If there is no space the size of the matrix `DoFMap0` is extended.

Figure 3-7 shows the history of the front size in two curves. The blue line represents the actual size of the front which shows that the maximum merged degrees of freedom in the matrix `DoFMap0` is the maximum front size 1473. The green line represents the number of non-trivial degrees of freedom in the front which shows the minimum number of degrees of freedom remaining in the front while the elements are merged. Release the degrees of freedom for each element from the matrix `DoFMap` gives the minimum degrees of freedom in the matrix `DoFMap0`.

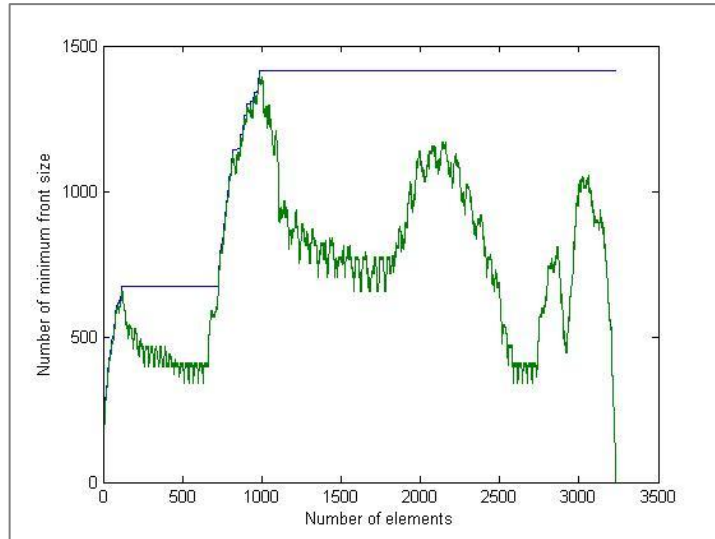


Figure 3-7 : History of the minimum front size

Table 3-3 shows the first 10 merged elements. The 1st merged element had 60 degrees of freedom. The reduced degrees of freedom are the number of degrees of freedom which can be removed during the merging. The remaining or front degrees of freedom are the number of degrees of freedom which remain during each element merging.

Element no.	Merged DoFs	Reduced DoFs	(Remaining) Front DoFs
1	60	12	48
2	84	12	72
3	108	12	96
4	132	12	120
5	156	12	144
6	180	12	168
7	189	12	177
8	198	12	186
9	222	12	210
10	231	12	219

Table 3-3 : Minimum front size

3.4.5 *Element matrices*

The element stiffness and mass matrices are calculated using equations (3.36) and (3.37). Gauss integration point coordinates and weight factors in Table 3-1 are used to evaluate stiffness \mathbf{K}_e and mass \mathbf{M}_e matrices for each element. The Gauss integration points $\{\eta_i, \xi_j, \zeta_k\}$ and weight coefficients $\{w_i, w_j, w_k\}$ are arranged in 64 entries. The element matrices are prepared depending on the element type and material properties for each element and the node point coordinates. The element model matrices \mathbf{K}_e and \mathbf{M}_e have dimensions (60×60).

3.4.6 *Assembling the element matrices*

The element matrices are merged from the individual element matrices into global matrices. The number of degrees of freedom of the global matrices MNDoF is the maximum front size. The elements are arranged in an order related to the global original point in the structure. The location for all degrees of freedom in each element called *frow* is identified. The element matrices are located at *frow* while the element matrices are merged into global matrices.

3.4.6.1 *Model reduction during merge*

The number of degrees of freedom is reduced using Guyan reduction during the merge. The reduced stiffness and mass matrices depend on the front size and element degrees of freedom. The slave degrees of freedom are chosen as the degrees of freedom with the maximum value of the diagonal $\mathbf{K}_{red}/\mathbf{M}_{red}$. The number of slave

degrees of freedom is the number of reduced degrees of freedom in the front for each element.

3.4.7 Eigenvalues and eigenvectors

The fourth MATLAB script called FE_MODEL loads up the reduced matrices \mathbf{K}_{red} and \mathbf{M}_{red} in a file called model_2. In this script, the eigenvalues with the associated eigenvectors for the reduced system are computed. The reduced modal vectors are expanded to full length modal vectors of the system.

3.5 Modes of Vibration

The first 20 eigenvalues of the computed finite element mode shapes are chosen. The results show that the first six natural frequencies represent rigid body modes. Table 3-4 shows the natural frequencies for modes of vibration. Wing first mode of vibration occurred at 13.44 Hz and has been shown in Figure 3-8. At this frequency the wing is bending symmetric moment. Wing anti-symmetric first mode of vibration occurs at 29.98 Hz shown in Figure 3-9. The wing torsion happens at frequencies 34.76Hz, 51.82Hz and 56.90Hz as in Figure 3-10, Figure 3-13 and Figure 3-14. The vertical first mode occurs at 38.41 Hz as in Figure 3-11. The vertical tail torsion happens at 88.89Hz and 147.07Hz as in Figure 3-16, Figure 3-20. The wing third mode of vibration and torsion happens at 64.98 Hz. The fuselage first mode of vibration occurs at a 60.98 Hz as shown in Figure 3-15. The mode of vibration of the wing tips and horizontal tail falls between the rigid mode and bending.

Mode no.	Natural Freq. HZ	Modes of Vibration
7	13.73	Wing first mode of vibration.
8	29.98	Wing anti-symmetric first mode of vibration.
9	34.76	Wing twist and wing tips first mode of vibration.
10	38.41	Wing anti-symmetric first mode of vibration, vertical tail first mode of vibration and horizontal tail bending.
11	48.57	Wing symmetric second mode of vibration.
12	51.82	Wing torsion and wing tips first mode of vibration.
13	56.90	Wing symmetric torsion.
14	60.40	Wing anti-symmetric second mode of vibration, fuselage first mode and vertical tail second mode.
15	64.98	Wing third mode of vibration and torsion, vertical tail first mode and wing tips symmetric first mode.
16	88.89	Wing anti-symmetric third mode of vibration, wing tips symmetric first mode and vertical tail twist.
17	107.50	Wing anti-symmetric third mode of vibration and wing tips symmetric first mode.
18	117.50	Wing anti-symmetric third mode of vibration, wing tips vertical tail twist.
19	136.49	Wing symmetric third mode and wing tips symmetric first modes of vibration.
20	147.07	Wing symmetric third mode of vibration, wing tips first mode vertical tail twist.

Table 3-4 : Natural frequencies and computed mode shapes

3.6 Conclusion

This chapter has identified structural models and the dynamic behaviour for GARTEUR-like aircraft structure. The presented work has introduced in 3-Dimensional 20-noded quadratic hexahedron (brick) element. The 20 modes of vibration and their frequencies are calculated. The first six modes represent rigid body modes. The number of degrees of freedom of global matrices (stiffness and mass matrices), which depends on the maximum front size, is reduced using Guyan reduction during the merge.

The following chapter presents an experimental modal analysis for GARTEUR-like aircraft structure using ground vibration testing GVT. The computed results obtained from finite element analysis will be compared with the experimental results obtained from GVT. Regarding reducing the finite element degrees of freedom, Craig Bampton transformations will be developed in chapter 9.

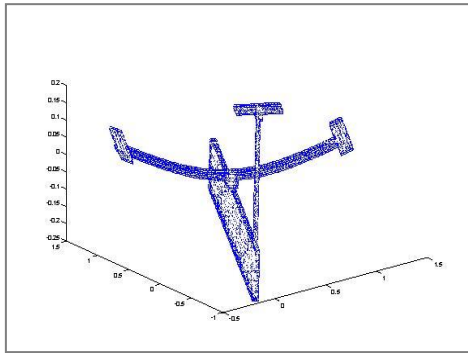


Figure 3-8 : Mode (7) Freq. 13.44 Hz

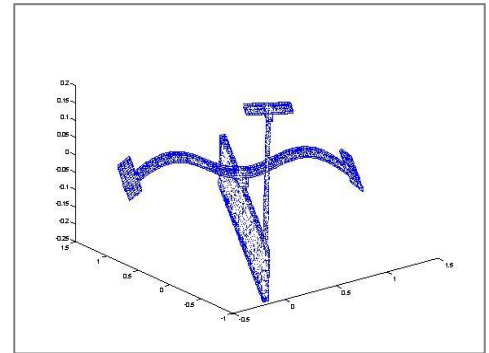


Figure 3-12 : Mode (11) Freq. 48.57 Hz

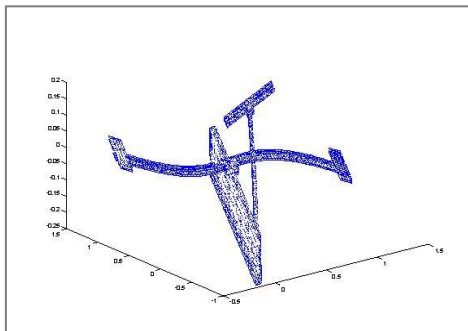


Figure 3-9 : Mode (8) Freq. 29.98 Hz

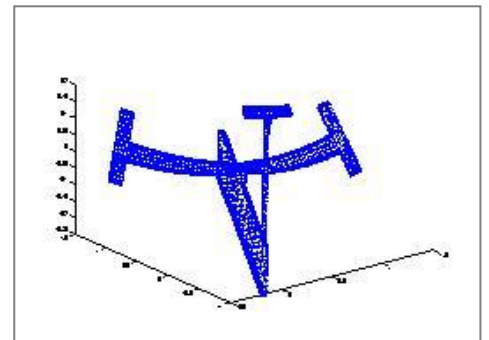


Figure 3-13 : Mode (12) Freq. 51.82 Hz

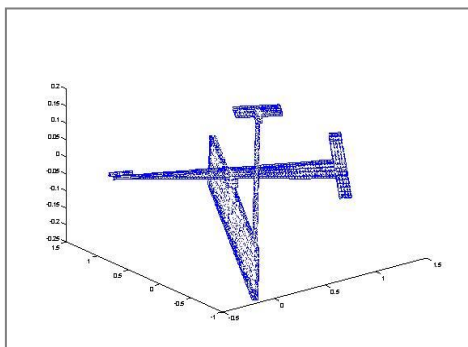


Figure 3-10 : Mode (9) Freq. 34.76 Hz

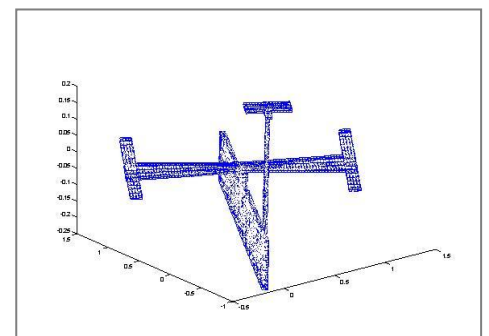


Figure 3-14 : Mode (13) Freq. 56.90 Hz

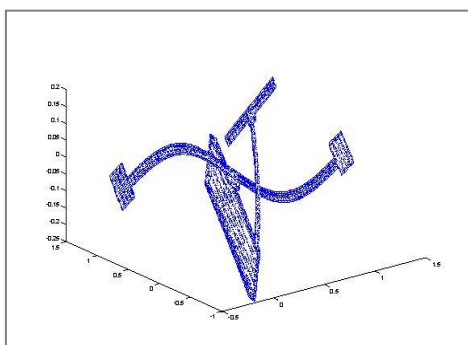


Figure 3-11 : Mode (10) Freq. 38.41 Hz

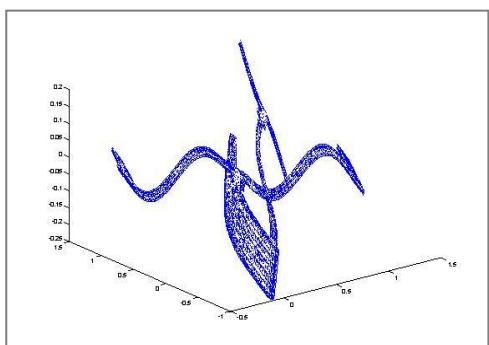


Figure 3-15 : Mode (14) Freq. 60.40 Hz

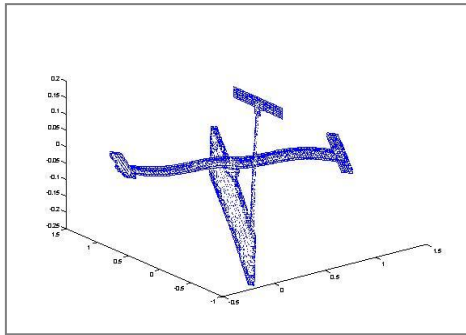


Figure 3-16 : Mode (16) Freq. 88.89 Hz

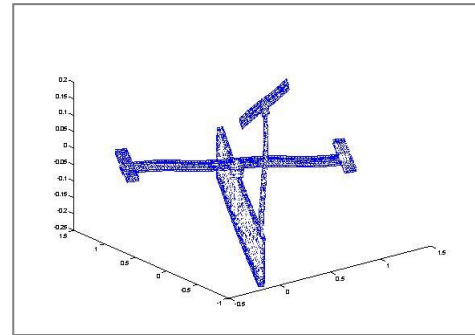


Figure 3-19 : Mode (19) Freq. 136.49 Hz

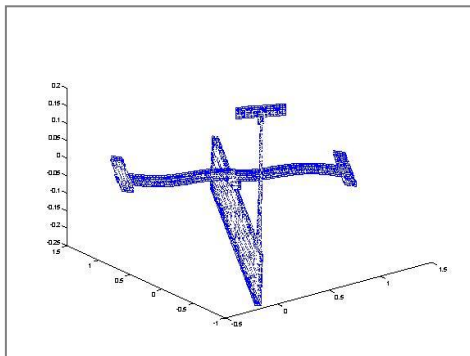


Figure 3-17 : Mode (17) Freq. 107.50 Hz

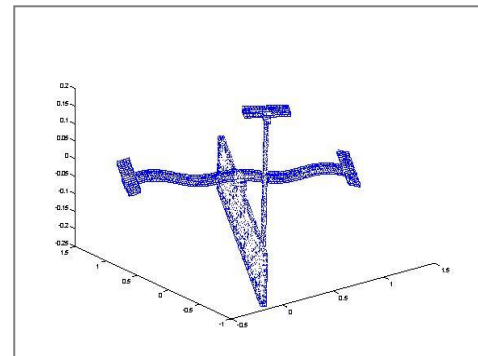


Figure 3-20 : Mode (20) Freq. 147.07 Hz

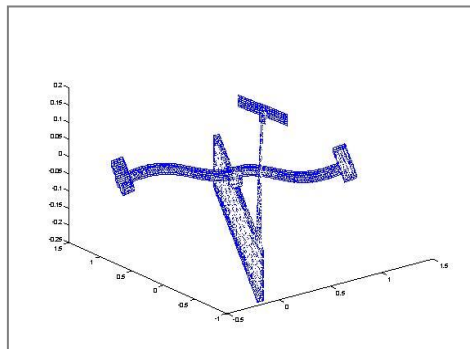


Figure 3-18 : Mode (18) Freq. 117.50 Hz

CHAPTER 4. Modal Testing

This chapter provides some background regarding the modal testing techniques and describes the methods that were used in this work for experimental modal analysis. In this chapter, the experimental data obtained from GVT will compare with the computed data obtained from the finite element analysis in the previous chapter.

A modal test is where a structure is mounted with controlled boundary conditions and vibrated with a known or measured excitation. This experimental test is used in order to obtain or update a mathematical model of the structure, determining the nature and extent of vibration response levels, and measurement of the dynamic properties. These properties are used to compare with corresponding data produced by a finite element or other theoretical model. In addition, they are used to validate the theoretical model prior to its use for predicting response levels to complex excitations. The corroboration (verification) of the tests of major modes of vibration can provide support of the basic validity of the theoretical model [10].

A technique is investigated in this study to extract modal data which are identified from a single column of the frequency response matrix. This technique overcomes the difficulty due to the conventional methods which require a series of measured FRFs at different points of excitation. This chapter also covers some general information about modal testing.

In this chapter, the modal analysis has been carried out on the “GARTEUR-like” structure (Figure 4-1). The model material used is Aluminium. The length of the

airplane is 1.5 m, and the wing span is 2.0 m with thickness 10mm. The presented work has been divided into two parts. The first part is obtaining the frequency response function using ground vibration testing (GVT). The second part focuses on extracting modal information using a single column of the frequency response function.

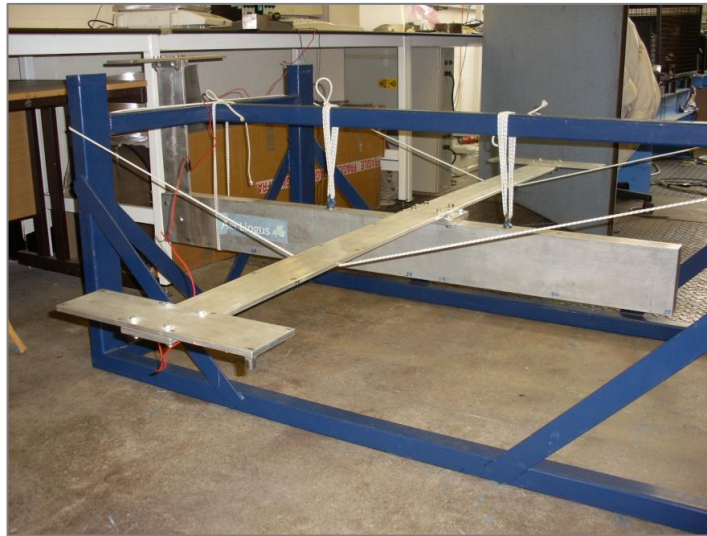


Figure 4-1 : GARTEUR-like aircraft structure

4.1 General Outline of Modal Testing

The most important task of modal testing is the preparation of the test structure itself. The first decision which has to be taken is whether the structure is to be tested with a free boundary condition or grounded. Free support means that the test structure is not attached to ground at any of its coordinates. In this condition, the structure will display rigid body modes in which there is no bending or other deformation. Due to errors in the measurements and difficulties in the simulation of free-free motion, the measured flexible mode shapes in general will not be orthogonal to the known rigid body modes [94].

Modal testing may involve the complete aircraft mounted on a soft support system to simulate free-free conditions, resting on its landing gear, specific parts on the complete airplane, or components, such as landing gear doors or control surfaces. The soft support systems are used to simulate a free-free condition and to reduce the frequency of the rigid-body modes.

4.1.1 Measurement setup

The main elements of equipment used in the experimental setup are:

1. An excitation device used to vibrate the structure. The source of the excitation signal which depends on the type of modal testing (sinusoidal from an oscillator, random from a noise generator and transient from a special pulse by applying an impact with a hammer). An amplifier selected to match the excitation device to vibrate the structure.
2. Transducers used to measure the excitation forces and the various responses of interest shown in Figure 4-2.
3. A spectrum analyser used to measure the various signals developed by the transducers in order to determine the magnitudes of the excitation forces and responses.



Figure 4-2 : Transducers to measure forces and responses

4.1.2 Excitation Methods

Various devices are available for exciting the structure. These can be divided into two types, contact and non-contact excitation. In contact excitation, the structure can be driven by either an attached shaker or an impulse. The transducers which are connected to the excitation devices are used to measure the excitation forces. Other possible sources of excitation include step relaxation (releasing from a deflected position) and ambient excitation (such as waves, wind or roadway excitation). These are special cases which are only used when the conventional methods are not possible. The non-contact excitation includes devices such as non contacting electromagnetic exciters and pneumatic acoustic excitation.

4.1.3 Response measurements

There are several methods for measuring the response by contact or non-contact response transducers such as piezoelectric accelerometers see Figure 4-3. Certain

characteristics normally govern the choice of accelerometers for structural testing. Structural elements must not be loaded by heavy accelerometers. Large structures may require high sensitivity accelerometers. Frequency range is not normally a problem, the upper frequency limit of most accelerometers exceeds the maximum frequency requirements of structural testing.

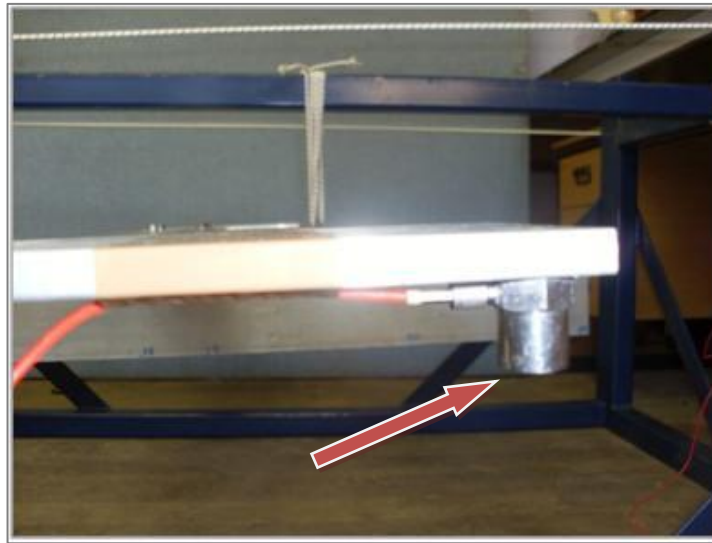


Figure 4-3 : Piezoelectric accelerometers

Two types of Laser Doppler Vibrometer (LDV) have become commonplace in modal testing. The standard single-point version and the scanning version (SLDV) have been used as non-contact transducers to measure response.

4.1.4 Obtaining Frequency Response Function (FRFs)

In modal testing, the inputs (usually forces) and outputs (usually accelerations) are functions of time. The properties of the structure itself create a relationship between the inputs and outputs and a key stage in modal testing is to determine that relationship. For linear systems the relationship between inputs and outputs is a

direct linear and frequency domain function where it can be expressed as a ratio of two polynomials. This ratio is called the frequency response function (FRF) which defines the ratio between the output response signals and the input force signals. The displacement response to force is *Receptance*. The velocity response is called *Mobility*. The acceleration response is called *Inertance* or *Accelerance*.

It is possible to derive frequency response function properties from excitation and response measurements when the vibration is periodic. Fourier series components of both the input force signal and the output response signals can be determined over the same range of frequencies. Once these two series are obtained, the FRF can be defined by computing the ratio of the response component to the input component for each discrete frequency.

The calculation of the Fourier transforms of both the excitation and response signals is required to obtain FRF properties from measurements made during the transient vibration test. The transient input and response signals are used to obtain an input/output function in the frequency domain. The ratio of these two functions can be computed to obtain an expression for the corresponding frequency response function:

$$\mathbf{H}(\omega) = \frac{\mathbf{y}(\omega)}{\mathbf{u}(\omega)} \quad (4.1)$$

As stated previously, the excitation may produce periodic, transient or pseudo-random signals. Fourier analysis is used to convert the signals from the time domain to the frequency domain.

The discrete Fourier transform or series is computed for both the force and response signals and the ratio of these transforms gives the FRF. There are two sets of parameters used to describe random signals; one based on the time domain is called the correlation function and the other in the frequency domain is called the spectral densities.

It is possible to determine FRF properties from the measurement and analysis of a random vibration test. The Fourier transform for the cross correlation function $R_{qf}(\tau)$ is used to produce the cross spectral density function $S_{qf}(\omega)$. The following formulas define the correlation functions and spectral density functions [10].

$$\begin{aligned} R_{ff}(\tau) &= E[f(t) \cdot f(t+\tau)] \\ S_{ff}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{ff}(\tau) e^{-i\omega\tau} d\tau \end{aligned} \quad (4.2)$$

$$\begin{aligned} R_{fq}(\tau) &= E[f(t) \cdot q(t+\tau)] \\ S_{fq}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{fq}(\tau) e^{-i\omega\tau} d\tau \end{aligned} \quad (4.3)$$

$$\begin{aligned} R_{qf}(\tau) &= E[q(t) \cdot f(t+\tau)] \\ S_{qf}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{qf}(\tau) e^{-i\omega\tau} d\tau \end{aligned} \quad (4.4)$$

$$\begin{aligned} R_{qq}(\tau) &= E[q(t) \cdot q(t+\tau)] \\ S_{qq}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{qq}(\tau) e^{-i\omega\tau} d\tau \end{aligned} \quad (4.5)$$

where $f(t)$ and $q(t)$ are input and response functions in the time domain respectively. Autocorrelation functions, $R_{ff}(\tau)$ and $R_{qq}(\tau)$, define as the expected value of the product two functions which are real functions and computed along the time axis. $S_{ff}(\omega)$, $S_{qq}(\omega)$ define the Auto-spectral densities which are real functions

of frequency. $R_{qf}(\tau)$ and $R_{fq}(\tau)$ define cross-correlation functions which are real functions of time, $S_{qf}(\omega)$ and $S_{fq}(\omega)$ define cross-spectral density which are real functions of frequency [10]. The frequency response function FRF can be determined from the spectral densities formula

$$\mathbf{H}_1(\omega) = \frac{S_{fq}(\omega)}{S_{ff}(\omega)} \quad (4.6)$$

$$\mathbf{H}_2(\omega) = \frac{S_{qq}(\omega)}{S_{qf}(\omega)} \quad (4.7)$$

Noise is expected to contribute to the input force signal and output response signals. The ratio between $\mathbf{H}_1(\omega)$ and $\mathbf{H}_2(\omega)$ is called the Coherence \mathfrak{C}^2 . The coherence can be shown to be less than or equal to 1.0. The coherence less than 1.0 represents the noise. A low coherence can occur in measurements near resonance and anti-resonance on lightly-damped structures. The reason for the low coherence in some cases is due to random noise.

$$\mathfrak{C}^2 = \frac{\mathbf{H}_1(\omega)}{\mathbf{H}_2(\omega)} \quad (4.8)$$

It is necessary to perform an averaging of several measurements of FRF estimates and coherence to establish confidence in results. The major advantage of employing a number of averages is to remove noise on the signals. The low coherence can be eliminated by taking many averages if the reason is random noise. Figure 4-4 shows frequency response function in the upper curve and the coherence in the lower curve.

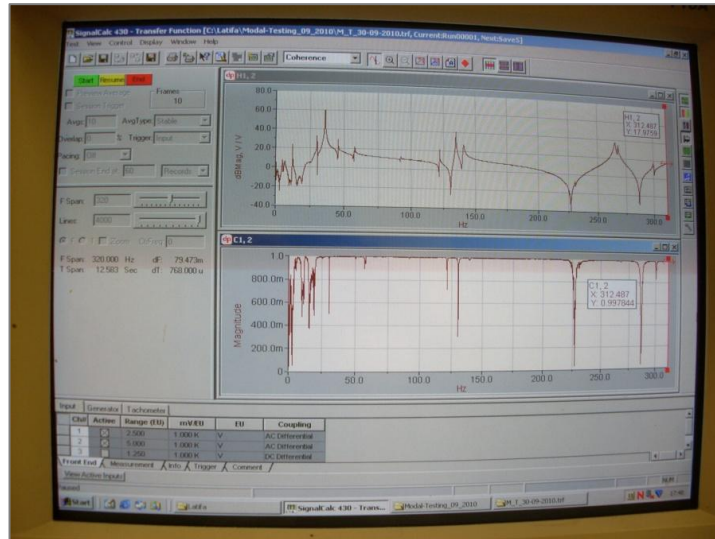


Figure 4-4 : FRF and coherence

4.2 Work done on the GARTEUR-like aircraft structure

The experimental work was conducted in vibration laboratory. The model structure shown in Figure 4-1 was suspended by set of elastic cords (bungees) to simulate free-free conditions and to reduce the frequency of the six rigid-body support modes as much as possible. The Aircraft-like structure is marked at 52 points (Figure 4-5) where it is excited using an impact hammer at each degree of freedom and the accelerometer response measurement is positioned at the left wing tip (point 1). Figure 4-6 shows the accelerometer location of the Aircraft-like structure.

The instrumentation set-up comprises are shown in Figure 4-7;

1. 2 charge Amplifier (one for input and one for output),
2. Force transducer (Hammer),
3. Accelerometer,

4. Oscilloscope,
5. PC driving spectrum Analyzer card.

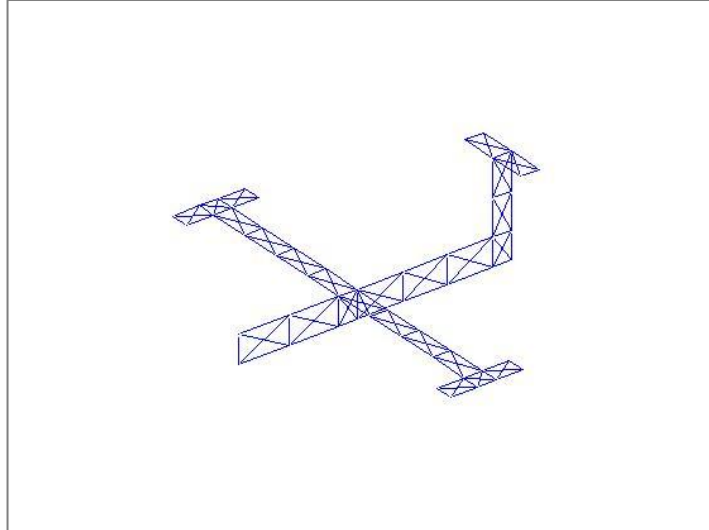


Figure 4-5 : Model structure

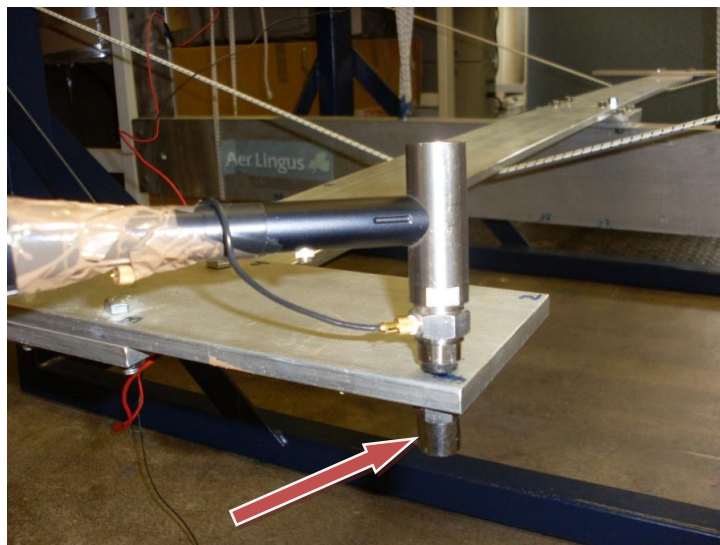


Figure 4-6 : Location of the accelerometer on airplane structure

The frequency range analysed spans 0 to 320 Hz and is divided into 4000 lines. The average of the response from 10 triggers is taken for each response point. MATLAB files are used to save the data such as transfer function H1_2 and coherence function

C1_2. Classical damping is assumed in this test. All the FRFs $\mathbf{H}(\omega)$ for 52 points have been obtained using one fixed accelerometer at wing tip shown in Figure 4-6.

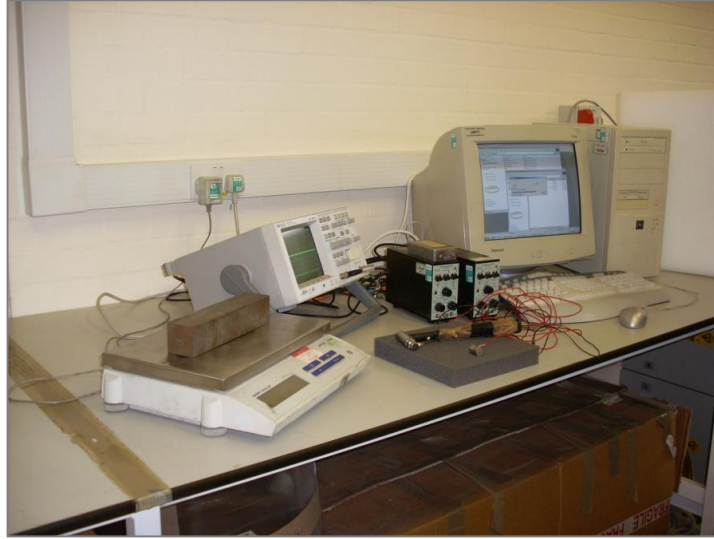


Figure 4-7 : Equipments used in modal testing

4.3 Calibration of a piezoelectric accelerometer

Calibration of a piezoelectric accelerometer was carried out on a mass weighted 4.5135 kg as shown in Figure 4-8. The mass is suspended to simulate free-free condition and excited using impact hammer as shown in Figure 4-9. We consider that the mass is a linear system of one degree of freedom as shown Figure 4-10. The equation of motion for linear system can be expressed as

$$\begin{aligned} \mathbf{f} &= \mathbf{S}_L \mathbf{u} \\ m\ddot{\mathbf{q}}(t) + d\dot{\mathbf{q}}(t) + k\mathbf{q}(t) &= \mathbf{f}(t) \\ \mathbf{y} &= \mathbf{S}_R^T \mathbf{q} \end{aligned} \quad (4.9)$$

The solution of equation (4.9) has two parts. Homogeneous solution, if $\mathbf{f}(t) = 0$, the solution of the homogeneous equation corresponds physically to that of free damped

vibration. Particular solution, if $\mathbf{f}(t) \neq 0$, the solution is due to the excitation irrespective of the homogeneous solution. The solution is assumed as

$$\mathbf{q}(t) = \mathbf{q}e^{i\omega t}, \quad \mathbf{f}(t) = \mathbf{f}e^{i\omega t} \quad (4.10)$$

Substituting equation (4.10) into equation (4.9) yields

$$\begin{aligned} \mathbf{f} &= \mathbf{S}_L \mathbf{u} \\ (\mathbf{k} - \omega^2 \mathbf{m} + i\omega \mathbf{d}) \mathbf{q} &= \mathbf{f} \\ \mathbf{y} &= \mathbf{S}_R^T \mathbf{q} \end{aligned} \quad (4.11)$$

Equation (4.11) can be expressed by unit mass as

$$\begin{aligned} \mathbf{f} &= \mathbf{S}_L \mathbf{u} \\ (\omega_n^2 - \omega^2 + i2\zeta\omega_n\omega) \mathbf{q} &= \mathbf{m}^{-1} \mathbf{f} \\ \mathbf{y} &= \mathbf{S}_R^T \mathbf{q} \end{aligned} \quad (4.12)$$

where $\omega_n^2 = \mathbf{m}^{-1} \mathbf{k}$, ω_n is natural frequency and $2\zeta\omega_n = \mathbf{m}^{-1} \mathbf{d}$, ζ is damping factor.

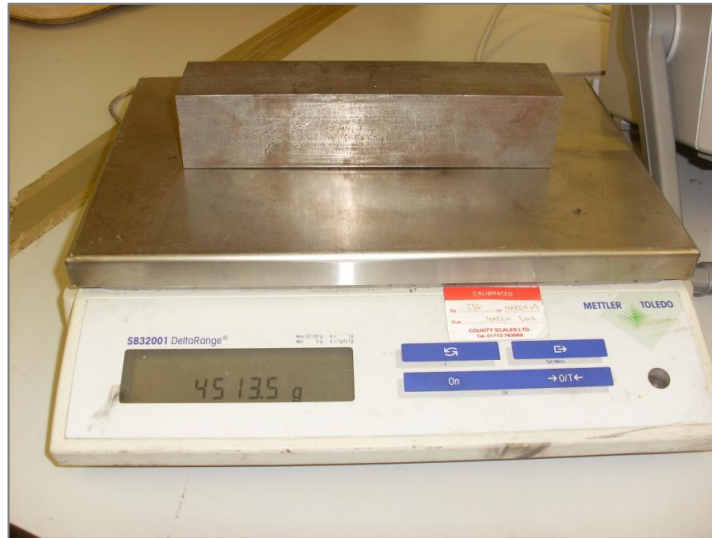


Figure 4-8 : Mass weighted



Figure 4-9 : Model of calibration

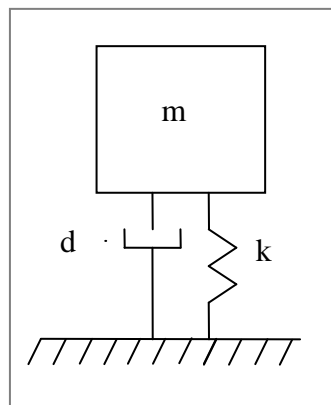


Figure 4-10 : Single degree of freedom system

4.3.1 Frequency response function (FRF) of linear system

Frequency response function represents a direct linear relationship between the input and output. The frequency response function at low frequency range indicates that the ratio between input and output signals is constant. This relationship is represented by the block diagram of a linear system in Figure 1-5. The frequency response function can be writes as

$$\mathbf{H}(\omega) = \frac{\mathbf{q}}{\mathbf{f}} = (\mathbf{k} - \omega^2 \mathbf{m} + i \omega \mathbf{d})^{-1} \quad (4.13)$$

Also the frequency response function can be written as

$$\mathbf{H}(\omega) = \frac{\omega_n^2 \mathbf{q}}{\mathbf{f}} = \mathbf{m}^{-1} \left(\left(1 - \left(\frac{\omega}{\omega_n} \right)^2 \right) + i 2\zeta \left(\frac{\omega}{\omega_n} \right) \right)^{-1} \quad (4.14)$$

The mass “m” has a single resonant frequency which occurs in the higher frequency range. Equation (4.14) represents the frequency response function $\mathbf{H}(\omega)$ which can be used at high frequency. At lower frequency range, the calculated frequency response function can be expressed as

$$\mathbf{H}(\omega) \approx \frac{\omega_n^2 \mathbf{q}}{\mathbf{f}} = \mathbf{m}^{-1} \quad (4.15)$$

Equation (4.15) shows that the frequency response function at lower frequency range is equal to inverse of the mass (kg^{-1}).

4.3.2 Correction factor C_f

In this experiment, the input and output signals is measured in voltage range. Decibel is a unit of measurement which is used in vibration measurements. The Decibel is expressed in terms of logarithm of the output to the input signals.

$$dB = 20 \times \log_{10} \left(\frac{x_{out}}{x_{in}} \right) \quad (4.16)$$

Figure 4-11 shows a calibration curve of the piezoelectric accelerometer in frequency range (0-320Hz). The curve shows that the measured frequency response function $\mathbf{H}_m(\omega)$ at low frequency has constant value. The measured frequency response function shown in Figure 4-11 has non-dimension (dB).

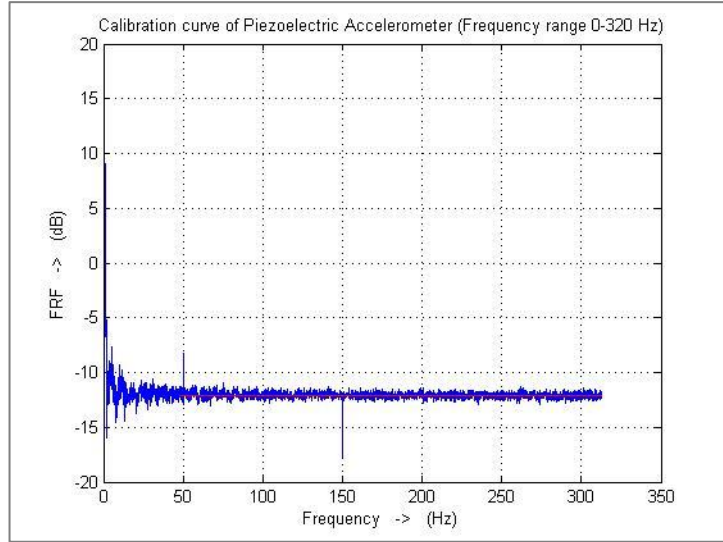


Figure 4-11 : Calibration curve of piezoelectric accelerometer

Figure 4-12 shows that the output accelerometer signal and the input force signal are converted through charge amplifier into voltage "V".

$$\mathbf{H}_m(\omega) = \left(\frac{a_c \times x_A}{\mathbf{f} \times x_F} \right) = \left(m^{-1} \frac{x_A}{x_F} \right) = m^{-1} C_f \quad (4.17)$$

The ratio $\left(\frac{x_A}{x_F} \right)$ represents the gain of output accelerometer to the gain of input force.

This ratio is called a correction factor C_f . In this ratio, the x_A has units $\mathbf{V}/(m/s^2)$, x_F has units $\mathbf{V}/N = \mathbf{V}/(kg.m/s^2)$ and the correction factor C_f has units $N/(m/s^2) = kg$.

It is easy to calculate the correction factor C_f for the simple mass using equation (4.15) and equation (4.17). The correction factor C_f is expressed as

$$C_f = \mathbf{H}_c(\omega)^{-1} 10^{(\mathbf{H}_m(\omega)/20)} = \mathbf{H}_c(\omega)^{-1} \mathbf{H}_r(\omega) \quad (4.18)$$

$$C_f = \mathbf{H}_c(\omega)^{-1} 10^{(\mathbf{H}_m(\omega)/20)} = \mathbf{H}_c(\omega)^{-1} \mathbf{H}_r(\omega) \quad (4.19)$$

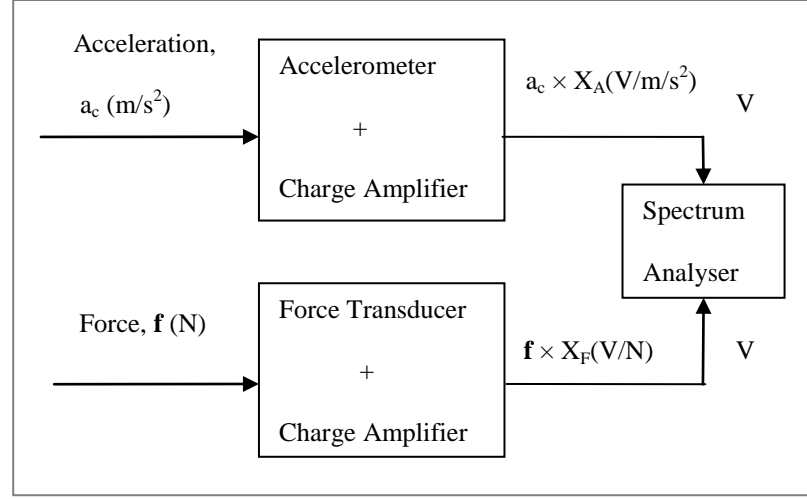


Figure 4-12 : Input and output signals diagram

Figure 4-13 shows the calibration curves of the piezoelectric accelerometer before and after modal testing of the aircraft structure in frequency range (0-80 Hz). The frequency response function before modal testing is $H_{mb} = -11.9852 \text{ dB}$, substituting into equation (4.18) yields

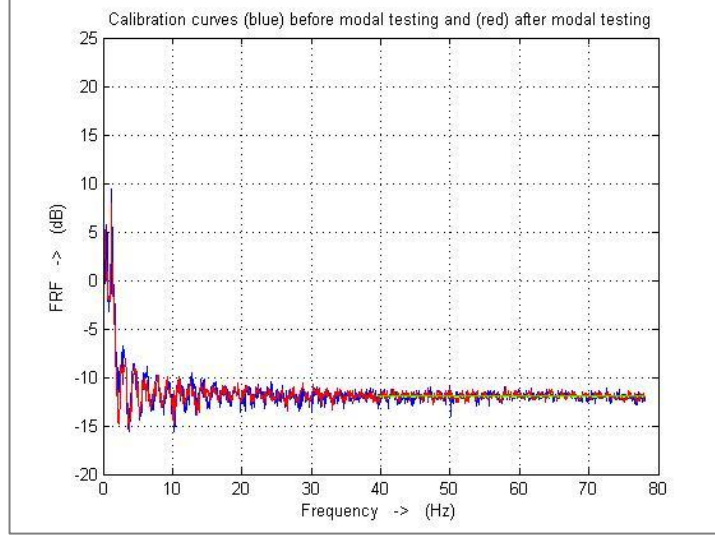
$$C_{fb} = \left(\frac{1}{4.5135} \right)^{-1} 10^{(-11.9852/20)} = 1.1357 \text{ kg} \quad (4.20)$$

The correction factor before starting modal testing of the GARTEUR-like aircraft structure is $C_{fb} = 1.1357 \text{ kg}$. However, the correction factor is changed after finishing the modal testing by 0.4336%. The measured frequency response function is $H_{ma} = -11.9099 \text{ dB}$ from Figure 4-13, substituting into equation (4.18) yields

$$C_{fa} = \left(\frac{1}{4.5135} \right)^{-1} 10^{(-11.9099/20)} = 1.1456 \text{ kg} \quad (4.21)$$

The correction factor after the modal testing of the GARTEUR-like aircraft structure is $C_{fa} = 1.1456 \text{ kg}$. The average of the correction factor is $C_f = 1.1406 \text{ kg}$. The correction factor C_f is used to correct the measured frequency response functions of the aircraft structure in all frequency range. All measured frequency response

functions of the aircraft structure are corrected by dividing by the correction factor C_f .



**Figure 4-13 : Calibration curves of piezoelectric accelerometer
before and after modal testing of aircraft structure**

4.4 Extracting modal information

The complete dynamic model of the second order system can be described by

$$\begin{aligned} \mathbf{f} &= \mathbf{S}_L \mathbf{u} \\ \mathbf{M} \ddot{\mathbf{q}}(t) + \mathbf{D} \dot{\mathbf{q}}(t) + \mathbf{K} \mathbf{q}(t) &= \mathbf{f}(t) \\ \mathbf{y} &= \mathbf{S}_R^T \mathbf{q} \end{aligned} \quad (4.22)$$

If this system is classically damped, then it is possible to define coordinate transformations $\{\Phi_L, \Phi_R\}$. These transformations transform the system matrices $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ into diagonal matrices $\{\Lambda, \Gamma, \mathbf{I}\}$ as in equations (2.15), (2.16) and (2.17) provided that \mathbf{M} is non-singular. All of the matrices here are real-valued.

The frequency response matrix, $\mathbf{H}(\omega)$ is $(n_o \times n_i)$, where n_o represents the number of outputs (the dimension of \mathbf{y}) and where n_i represents the number of inputs (the dimension of \mathbf{u}). $\mathbf{H}(\omega)$ can be expressed in terms of either the original representation equation (2.39) or the transformed modal representation equation (2.41) thus

$$\begin{aligned}\mathbf{H}(\omega) &= \mathbf{S}_R^T (\mathbf{K} + i\omega\mathbf{D} - \omega^2\mathbf{M})^{-1} \mathbf{S}_L \\ \mathbf{H}(\omega) &= \tilde{\Phi}_R (\Lambda + i\omega\Gamma - \omega^2\mathbf{I})^{-1} \tilde{\Phi}_L^T\end{aligned}\tag{4.23}$$

where $\tilde{\Phi}_R = \mathbf{S}_R^T \Phi_R$ and $\tilde{\Phi}_L^T = \Phi_L^T \mathbf{S}_L$.

Modal information can be extracted from the measured frequency response function $\mathbf{H}(\omega)$. A single entry of the frequency response matrix can be constructed as

$$h_{ij} = \sum_{k=1}^n \frac{\tilde{\Phi}_{R,ik} \tilde{\Phi}_{L,jk}}{(\lambda_{kk} + i\omega\gamma_{kk} - \omega^2)}\tag{4.24}$$

$\tilde{\Phi}_{L,jk}$ represents the amount by which one unit of excitation at input degree of freedom j excites mode k . $\tilde{\Phi}_{R,ik}$ represents the amount of output at output degree of freedom i which is contributed by each mode k . Extracting the single mode $\tilde{\Phi}_{R,ik}$ of the modal matrix $\tilde{\Phi}_R$ requires a series of measurements of FRF including the point FRF at the excitation position [10, 113].

4.4.1 A single column of the frequency Response- $\mathbf{g}(\omega)$

Consider that there is only one independent source of excitation. Then for some $(n_i \times 1)$ matrix \mathbf{p}

$$\mathbf{u}(t) = \mathbf{p}u(t) \quad \text{or} \quad \mathbf{u}(\omega) = \mathbf{p}u(\omega)\tag{4.25}$$

and in this case, we may define a single column of frequency response, $\mathbf{g}(\omega)$ according to

$$\begin{aligned}\mathbf{g}(\omega) &= \mathbf{H}(\omega)\mathbf{p} \\ \mathbf{g}(\omega) &= \mathbf{S}_R^T \left(\mathbf{K} + i\omega\mathbf{D} - \omega^2\mathbf{M} \right)^{-1} (\mathbf{S}_L\mathbf{p}) \\ \mathbf{g}(\omega) &= \tilde{\mathbf{\Phi}}_R \left(\mathbf{\Lambda} + i\omega\mathbf{\Gamma} - \omega^2\mathbf{I} \right)^{-1} (\tilde{\mathbf{\Phi}}_L^T\mathbf{p})\end{aligned}\tag{4.26}$$

From this point, we will ignore the formulation of frequency response from the general system representation equation (2.39) and we concentrate specifically on the representation of equation (2.41). $(\tilde{\mathbf{\Phi}}_L^T\mathbf{p})$ is abbreviated as \mathbf{v} .

Define the general i^{th} entry $g_i(\omega)$ of frequency response $\mathbf{g}(\omega)$ and we then have

$$g_i(\omega) = \sum_{k=1}^n \frac{\phi_{R,ik} v_k}{(\lambda_{kk} + i\omega\gamma_{kk} - \omega^2)} = \sum_{k=1}^n \frac{w_{ik}}{(\lambda_{kk} + i\omega\gamma_{kk} - \omega^2)}\tag{4.27}$$

$w_{ik} = \phi_{R,ik} v_k$ is a real-valued scalar which is called a *modal participation factor*.

Specifically, it reflects the amount by which one unit of the single independent excitation source, $\mathbf{u}(t)$, causes a response at the i^{th} output degree of freedom through mode $\#k$. In modal analysis, only these *modal participation factors* can be extracted.

If we have a numerical model also in the form $\{\mathbf{K}, \mathbf{D}, \mathbf{M}, \mathbf{S}_L, \mathbf{S}_R\}$ from equation (2.39), then some more information can be extracted. (In fact, only $\{\mathbf{M}, \mathbf{S}_L, \mathbf{S}_R\}$ are needed).

The modal participation factors, w_{ik} can be assembled into a real matrix \mathbf{W} , and then $\mathbf{g}(\omega)$ can be written in this following very compact form as

$$\mathbf{g}(\omega) = \mathbf{W} \times \text{diag} \left(\left(\mathbf{\Lambda} + i\omega\mathbf{\Gamma} - \omega^2\mathbf{I} \right)^{-1} \right)\tag{4.28}$$

In equation (4.28), the $\text{diag}(\cdot)$ operator serves to convert a diagonal matrix back into a column having n entries where n represents the number of independent modes. It should be noted that if any two or more “different” modes happen to have identical

λ_{kk} and γ_{kk} , then it is always possible to write (4.28) using fewer columns in \mathbf{W} . A single-column of FRFs is never sufficient to discriminate between two such modes. If two different columns of FRFs have been acquired, this discrimination is possible between two modes having identical λ_{kk} and γ_{kk} .

4.4.2 *Extracting modes given $\{\Lambda, \Gamma, \mathbf{I}\}$*

Many different methods exist for modal analysis performed directly on the FRF curves. Modal analysis methods depend upon whether a single mode is to be extracted at a time or several modes at a time. Moreover, modal analysis methods divide into two types, the first type analyses single FRF curves at a time and the second type analyse several curves. There is no single method which is best for all cases. A full discussion of these methods can be found in [10].

In equation (4.28), $\mathbf{g}(\omega)$ is a vector function of frequency. The same is true of $\text{diag}\left(\left(\Lambda + i\omega\Gamma - \omega^2\mathbf{I}\right)^{-1}\right)$. From a modal test, we acquire (an approximation to) $\mathbf{g}(\omega)$ for a large number of discrete frequencies. These frequencies are usually equally spaced and usually extend from 0 Hz up to some maximum value but there is no particular requirement for this. These frequencies can be arranged to form the vector \mathbf{f}_r . By forming equation (4.28) for each one of these individual frequencies, we can gather the results together into a single matrix form thus

$$\mathbf{G} = \mathbf{W}\mathbf{A} \quad (4.29)$$

$$\mathbf{A} =: \text{diag}\left(\left(\Lambda + i\omega\Gamma - \omega^2\mathbf{I}\right)^{-1}\right) \quad (4.30)$$

Matrix \mathbf{G} has dimensions $(n_o \times n_f)$ where n_f is the number of frequencies at which the FRF has been collected. Matrix \mathbf{A} has dimensions $(n \times n_f)$ and matrix \mathbf{W} has dimensions $(n_o \times n)$. Since \mathbf{G} is known and \mathbf{A} can be constructed quite easily (assuming that $\{\Lambda, \Gamma\}$ and the actual values of the n_f different trial frequencies are known), it is clearly a linear problem to determine \mathbf{W} . Invariably, n_f is much greater than n and so the problem of finding \mathbf{W} from \mathbf{G} and \mathbf{A} is an over-determined one. Because \mathbf{W} is real, equation (4.29) can be broken into two parts:

$$\bar{\mathbf{G}} = [\text{Re}(\mathbf{G}) \quad \text{Im}(\mathbf{G})] = \mathbf{W} [\text{Re}(\mathbf{A}) \quad \text{Im}(\mathbf{A})] = \mathbf{W} \bar{\mathbf{A}} \quad (4.31)$$

and a good solution is found using the right pseudo-inverse of $\bar{\mathbf{A}}$.

$$\mathbf{W} \approx \left((\bar{\mathbf{G}} \bar{\mathbf{A}}^T) (\bar{\mathbf{A}} \bar{\mathbf{A}}^T)^{-1} \right) \quad (4.32)$$

Columns of matrix \mathbf{W} contain multiples of the (right) mode-shapes.

One other comment is necessary about equation (4.31). In effect, equation (4.32) says that \mathbf{W} is the real part of the product of $\bar{\mathbf{G}}$ multiplied by the real pseudo-inverse of $\bar{\mathbf{A}}$.

4.4.3 Iteratively adjusting $\{\Lambda, \Gamma\}$ to minimise vector of residuals Δr

Combining equations (4.29) and (4.31) leads to this definition of a matrix, \mathbf{R} of residuals

$$\mathbf{R} = \mathbf{G} - \left((\mathbf{G} \mathbf{A}^H) (\mathbf{A} \mathbf{A}^H)^{-1} \right) \mathbf{A} \quad (4.33)$$

$$\bar{\mathbf{R}} = [\text{Re}(\mathbf{R}) \quad \text{Im}(\mathbf{R})] = \bar{\mathbf{G}} - \left((\bar{\mathbf{G}} \bar{\mathbf{A}}^T) (\bar{\mathbf{A}} \bar{\mathbf{A}}^T)^{-1} \right) \bar{\mathbf{A}} \quad (4.34)$$

In general, \mathbf{R} is complex and $\bar{\mathbf{R}}$ will be real. Their dimensions are the same as the dimensions of \mathbf{G} ($n_o \times n_f$) and $\bar{\mathbf{G}}$ ($n_o \times 2n_f$) respectively. For reasons which become obvious shortly, it is appropriate for us to vectorise $\bar{\mathbf{R}}$ to form \mathbf{r} having dimension ($2n_on_f \times 1$).

$$\mathbf{r} = \text{vec}(\bar{\mathbf{R}}) \quad (4.35)$$

Now, since \mathbf{A} is a function of the three quantities $\{\mathbf{\Lambda}, \mathbf{\Gamma}, \mathbf{f}\}$, it is evident from equation (4.32) that $\bar{\mathbf{R}}$ is also a function of these three quantities and in view of equation (4.34), it is straightforward now to define a total error, E , according to

$$E = \mathbf{r}^T \mathbf{r} \quad (4.36)$$

Clearly, E is also a function of the three quantities $\{\mathbf{\Lambda}, \mathbf{\Gamma}, \mathbf{f}\}$ and it is a single real-valued overall measure of the poorness-of-fit.

It is convenient to put together all of the diagonal entries of $\{\mathbf{\Lambda}, \mathbf{\Gamma}\}$ into a single vector of $2n$ parameters denoted $\boldsymbol{\theta}$. Since we can consider that for any one set of FRFs, the frequencies in \mathbf{f}_r are fixed, we can assert that \mathbf{r} is a function of $\boldsymbol{\theta}$. Equations (4.32) and (4.34) together show how to form $\mathbf{r}(\boldsymbol{\theta})$ given $\boldsymbol{\theta}$. A single iterative step can be carried out which begins with a given vector of parameters, $\boldsymbol{\theta}_0$ and ends with an improved vector of parameters, $\boldsymbol{\theta}_1 = (\boldsymbol{\theta}_0 + \Delta\boldsymbol{\theta})$.

From $\boldsymbol{\theta}_0$, we calculate the corresponding $\mathbf{r}_0 = \mathbf{r}(\boldsymbol{\theta}_0)$ using equations (4.32) and (4.34). Subsequently, we determine the complex Sensitivity Matrix, \mathbf{S} which is defined by the equation

$$\mathbf{r}(\boldsymbol{\theta}_0 + \Delta\boldsymbol{\theta}) = \mathbf{r}_0 + \Delta\mathbf{r} \approx \mathbf{r}_0 + \mathbf{S}\Delta\boldsymbol{\theta} \quad (4.37)$$

It is possible to determine \mathbf{S} analytically and this is probably the fastest and most numerically stable means by which to obtain \mathbf{S} . The alternative is to determine \mathbf{S} numerically. At present, this numerical determination of \mathbf{S} is adopted (accepted) simply because it is expedient (convenient). A central-difference approach is used. For each one of the $2n$ entries of $\boldsymbol{\theta}$, a small positive change, δ , is made in the entry and \mathbf{r}_+ is computed. Then a small negative change, $-\delta$, is made in the same entry and \mathbf{r}_- is computed. Finally, the vector expression $((\mathbf{r}_+ - \mathbf{r}_-)/2\delta)$ provides one whole column of the matrix \mathbf{S} .

Having found \mathbf{S} and $\Delta\mathbf{r}$, a first approximation to the optimum adjustment, $\Delta\boldsymbol{\theta}$, is discovered as

$$\Delta\boldsymbol{\theta} = -\mathbf{S}^+ \Delta\mathbf{r} \quad (4.38)$$

This is only a first approximation because the linear approximation expressed by equation (4.37) may not apply accurately over the entire *optimum adjustment* put forward by equation (4.38). To make the iterative refinement of the parameters robust, it is necessary to combine a *line-search* strategy with equation (4.38). Equation (4.38) is used to determine the small adjustment in the parameters. The actual adjustment is $(x\Delta\boldsymbol{\theta})$ where x is a scalar—normally much less than 1. As the iterative procedure advances, x tends to become less and less and by the time that x has reached 0.001, the iteration is probably as close as it is going to get.

4.4.4 Obtaining initial values for the parameters $\{\Lambda, \Gamma\}$

It was stated above that we are supposing that the outcome from some modal testing is a single column $\mathbf{g}(\omega)$ having n_o entries in it. This outcome can be represented as the complex $(n_o \times n_f)$ matrix, \mathbf{G} .

The $\text{sum}(\text{abs}(\mathbf{G}))$ is a real-valued row-vector having n_f always-positive entries in it. The transpose of this row vector is denoted as \mathbf{z} . In this work, the interested frequency range has been chosen 25-175 Hz. The plot of $\log(\mathbf{z})$ with respect to the frequency range f is shown in Figure 4-14.

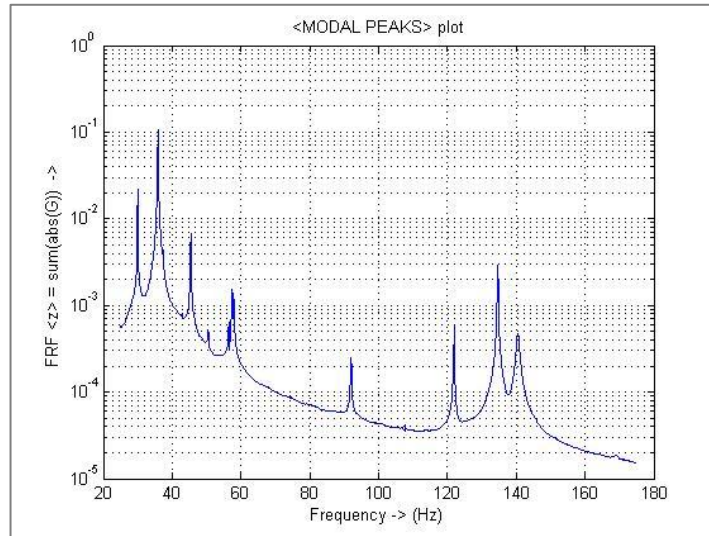


Figure 4-14 : Modal peaks

This plot is known in modal analysis as a *modal peaks* plot. If a significant peak in response has occurred in any one of the n_o FRFs obtained from the testing, this peak will appear as a significant peak in the *modal peaks* plot. From the modal peaks plot, numerous discrete frequency-bands are identified containing resonances indicating at the same time how many resonance frequencies there are within each one.

Figure 4-15 shows numerous discrete frequency-bands containing resonances. Each resonant frequency band is considered to contain one and only one resonant frequency and an initial estimate for this resonant frequency is provided at the same time as the upper and lower frequency limits of the band are specified. The frequency bands are usually relatively small. Typically, they might contain 5-12 different frequency lines each.

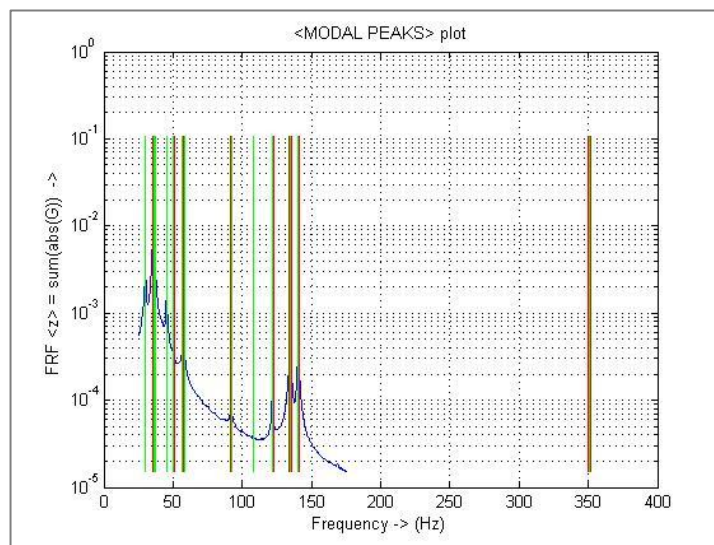


Figure 4-15 : Modal peaks / frequency-bands

Having identified a number of frequency bands (each containing a single resonance) and having speculated what the value of that resonant frequency is (by inspection also), an automatic procedure can be employed to produce an estimate of the damping factor associated with each individual resonance.

This procedure works on magnitudes only and seeks to minimise the difference between the normalised vector of magnitudes from the modal peaks plot and a normalised vector of magnitudes from an analytical single-degree-of-freedom model.

The nature of this procedure for estimating the damping associated with a given mode is very similar to the procedure described above for adjusting simultaneously the parameters for all modes. It involves a sensitivity matrix and a vector of residuals but there is only one single parameter at a time being adjusted in this present case. Figure 4-16 shows the error (square of residuals) vs. the log of damping factor.

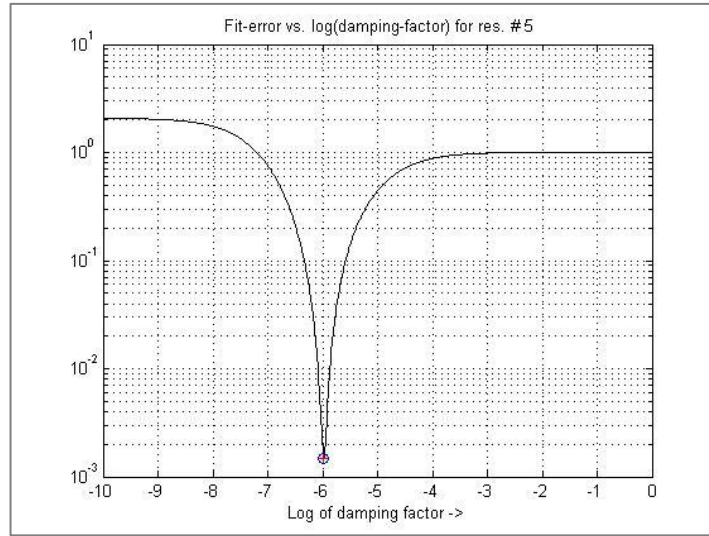


Figure 4-16 : Error vs. Log of damping factor

The reconstructed modal peaks which are shown in Figure 4-17 can be calculated from equation (4.29). The errors between the original and reconstructed modal peaks can be calculated from the sum of squares of the residual $\bar{\mathbf{R}}$ which is $sum(sqrt(\bar{\mathbf{R}})) = 1.36e-005$.

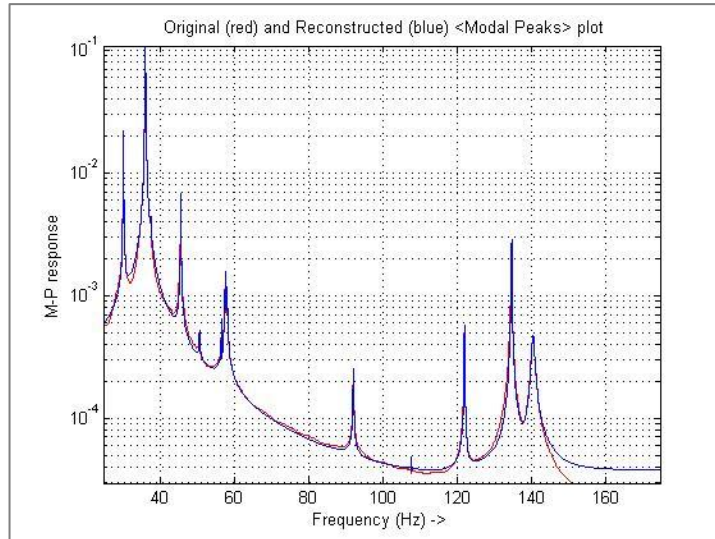


Figure 4-17 : Reconstructed modal peaks

4.5 Modes of vibration

Table 4-1 shows experimental results (natural frequencies and mode shapes) which have been extracted at each modal peak. The range of frequency has been chosen between 25-175 Hz. Figure 4-18 - Figure 4-30 shows that most of the modes in this range are wing modes. The wing first mode of vibration occurs at 29.96 Hz, and the horizontal tail first mode of vibration happens at 92.01 Hz as shown in Figure 4-23, Figure 4-24 and Figure 4-26. Figure 4-28 shows that fuselage first mode of vibration occurs at 121.98 Hz, that. The mode of vibration of the wing tips occurs at 107.7 Hz.

Mode no.	Natural Freq. HZ	Mode of Vibration
1	29.96	Wing anti-symmetric first mode of vibration.
2	35.88	Wing torsion anti-symmetric.
3	37.32	Wing torsion symmetric and wing tips anti-symmetric first mode.
4	45.46	Wing symmetric first mode of vibration.
5	50.46	Wing second mode of vibration and torsion.
6	56.41	Wing anti-symmetric second mode of vibration and horizontal tail first mode.
7	57.47	Wing second anti-symmetric mode of vibration.
8	57.90	Wing anti-symmetric second mode of vibration and wing tips first mode.
9	92.01	Wing third mode of vibration and horizontal tail anti-symmetric first bending
10	107.70	Wing tips first bending anti-symmetric.
11	121.98	Wing anti-symmetric third mode of vibration and Fuselage first mode of vibration.
12	134.68	Wing third mode of vibration symmetric.
13	140.48	Wing third mode of vibration anti-symmetric and fuselage first mode.

Table 4-1 : Experimental natural frequencies and extracted modes

4.6 Comparison of modal data

Comparisons of experimental data from GVT, analytical data from FE and GERTEUR published data in reference [20] are shown in Table 4-2. The measured results are compared with the computed results obtained from the finite element analysis in the previous chapter. Table 4-2 shows the percentage errors between the measured data and the computed data. The maximum errors shown in the table is - 12.23 % at mode 9.

These discrepancies may happen due to using incorrect data in the FE model. The material properties used in finite element analysis might cause some errors and do not include damping. The experimental procedure used may also cause some errors. For example, simulating the aircraft structure to simulate free-free condition may cause some changes in the frequency.

The measured results also have been compared with experimental data for GARTEUR group [6, 20]. The measured frequency responses were differing slightly than from GARTEUR group structure. There are some reasons to identify the cause of these discrepancies. The modal testing has not been carried out on the same GARTEUR group structure. The GARTEUR group structure [20] weight is 44kg. The different weight between two structures may cause some difference in frequencies. Modal testing has been done using a single output (Accelerometer) whereas 24 reference accelerometer locations were used. The way of suspension of the structure may cause some discrepancies.

Mode no.	Comparison of natural frequency for GARTEUR-like aircraft structure					
	Exp.	F.E.	%	Exp.	Publications	%
1	29.96	29.98	-0.07	29.96	-	-
2	-	-	-	-	34.88	-
3	35.88	34.76	3.12	35.88	35.36	1.45
4	37.32	38.41	-2.92	37.32	36.71	1.65
5	45.46	48.57	-6.84	45.46	50.09	-10.19
6	50.46	51.82	-2.69	50.46	50.72	-0.52
7	56.41	56.9	-0.87	56.41	56.44	-0.05
8	57.47	60.40	-5.10	57.47	65.14	-13.35
9	57.90	64.98	-12.23	57.90	69.64	-20.28
10	92.01	88.89	3.39	92.01	-	-
11	107.70	107.50	0.19	107.70	105.47	2.07
12	121.98	117.50	3.67	121.98	-	
13	134.68	136.49	-1.34	134.68	134.68	0
14	140.48	147.07	-4.69	140.48	145.87	-3.84

Table 4-2 : Comparison of modal data from different tests

4.7 Conclusions

This chapter has identified the structural dynamic characteristics for GARTEUR-like aircraft structure. The experimental modal analysis was carried out on a physical implementation of the GARTEUR-like aircraft structure. This Chapter presented a method to assure that modal properties were identified from a single column of frequency response. The presented method focused on fitting the modal peak one by one. The modal properties: natural frequency and modal damping ratio have been identified for each single mode of the vibrating system. The method overcomes the difficulty of conventional methods that require a series of measured FRFs at different points of excitation.

The comparison between different results in Table 4-2 shows that the experimental results obtained from GVT do not match exactly the calculated results obtained from the finite element model. These discrepancies between the measured and computed data can be assessed. The following chapter presents methods to calculate the rate of change of eigenvalues and eigenvectors.

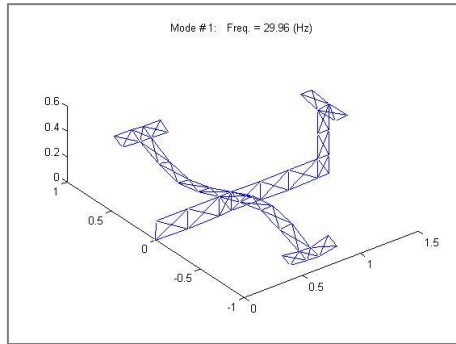


Figure 4-18 : Mode (1) Freq. 29.96 Hz

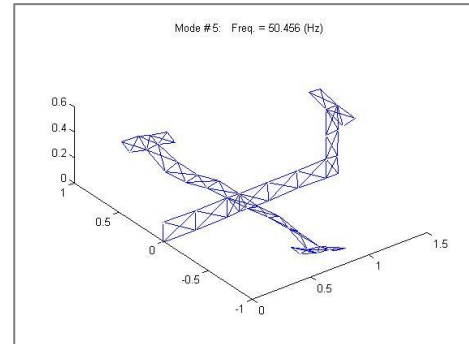


Figure 4-22 : Mode (5) Freq. 50.46 Hz

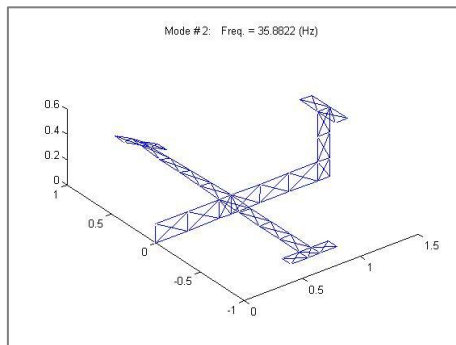


Figure 4-19 : Mode (2) Freq. 35.88 Hz

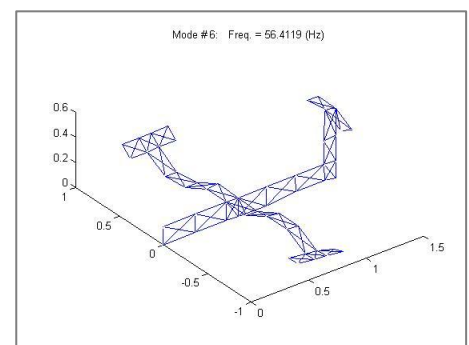


Figure 4-23 : Mode (6) Freq. 56.41 Hz

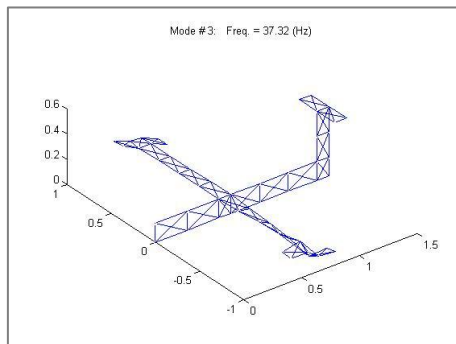


Figure 4-20 : Mode (3) Freq. 37.32 Hz

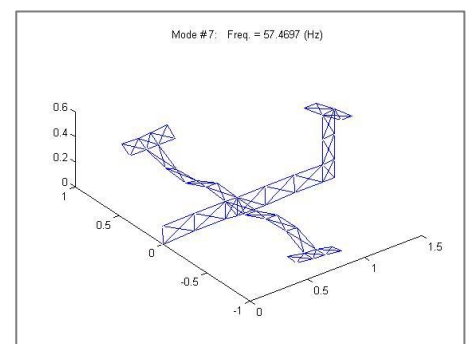


Figure 4-24 : Mode (7) Freq. 57.47 Hz

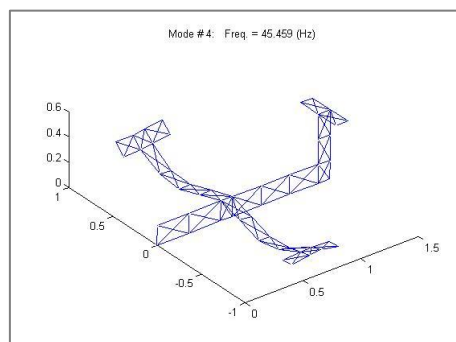


Figure 4-21 : Mode (4) Freq. 45.46 Hz

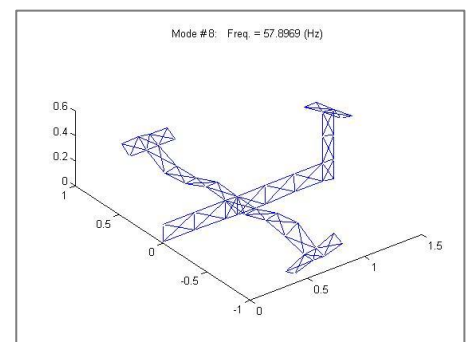


Figure 4-25 : Mode (8) Freq. 57.90 Hz

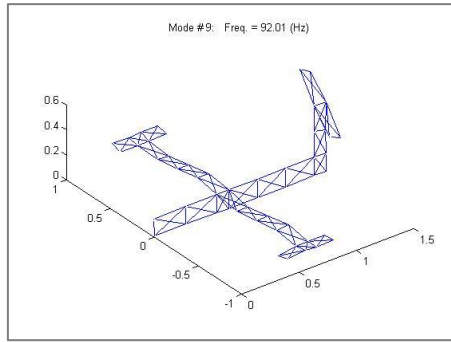


Figure 4-26 : Mode (9) Freq. 92.01 Hz

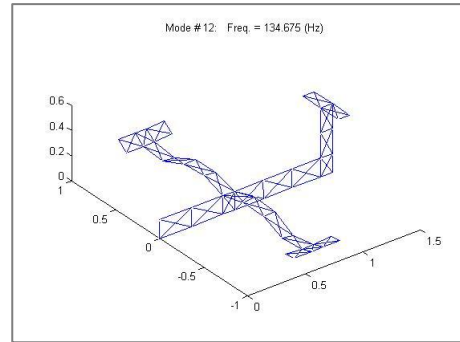


Figure 4-29 : Mode (12) Freq. 134.68 Hz

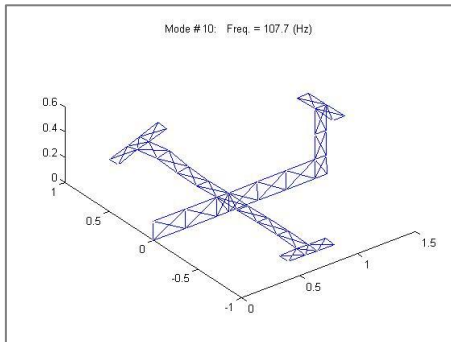


Figure 4-27 : Mode (10) Freq. 107.70 Hz

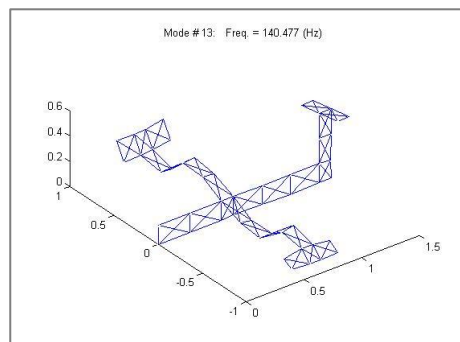


Figure 4-30 : Mode (13) Freq. 140.48 Hz

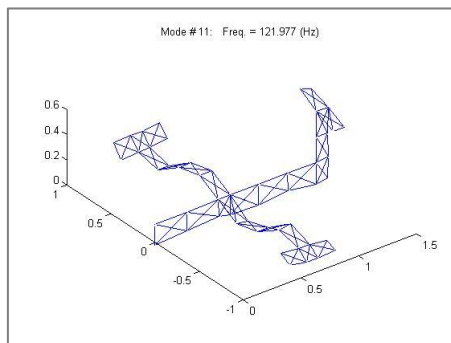


Figure 4-28 : Mode (11) Freq. 121.98 Hz

CHAPTER 5. Eigenvalue and Eigenvector Derivatives

The discrepancies between experimental results and analytical results in previous chapters can be estimated. This chapter investigates methods to calculate eigenvalue and eigenvector derivatives. These methods generalise the ideas implemented for undamped systems to generally damped systems using structure preserving equivalencies SPEs.

Dynamic analysis is required for design, modification and verification of the complex structures to satisfy dynamic response restrictions. It is common to compare the measured modal information $\{\Lambda_m, \Phi_{Lm}, \Phi_{Rm}\}$ -*eigenvalues and their corresponding left and right eigenvectors*-with the predicted modal information $\{\Lambda_a, \Phi_{La}, \Phi_{Ra}\}$. Changes in the system matrices $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ cause direct effects in the dynamic properties. The system matrices can be represented as

$$\mathbf{K}(\boldsymbol{\theta}) = \mathbf{K}_o + \boldsymbol{\theta}_1 \mathbf{K}_1 + \boldsymbol{\theta}_2 \mathbf{K}_2 + \dots + \boldsymbol{\theta}_p \mathbf{K}_p = \mathbf{K}_o + \sum_{j=1}^p \boldsymbol{\theta}_j \mathbf{K}_j \quad (5.1)$$

$$\mathbf{M}(\boldsymbol{\theta}) = \mathbf{M}_o + \boldsymbol{\theta}_1 \mathbf{M}_1 + \boldsymbol{\theta}_2 \mathbf{M}_2 + \dots + \boldsymbol{\theta}_p \mathbf{M}_p = \mathbf{M}_o + \sum_{j=1}^p \boldsymbol{\theta}_j \mathbf{M}_j \quad (5.2)$$

$$\mathbf{D}(\boldsymbol{\theta}) = \mathbf{D}_o + \boldsymbol{\theta}_1 \mathbf{D}_1 + \boldsymbol{\theta}_2 \mathbf{D}_2 + \dots + \boldsymbol{\theta}_p \mathbf{D}_p = \mathbf{D}_o + \sum_{j=1}^p \boldsymbol{\theta}_j \mathbf{D}_j \quad (5.3)$$

The variables $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \dots, \boldsymbol{\theta}_p)$ can be assembled into a vector $\boldsymbol{\theta}$ of unknown or uncertainty parameters. The subscript “ p ” is the number of entries in the vector $\boldsymbol{\theta}$ where $p < 3n^2$. Matching the discrepancies between the physical and computed modal

information can be achieved by computing the sensitivity of this modal information to changes in the parameters. This can be done either numerically or analytically

$$\lambda_i = \lambda_{oi} + \Delta \mathbf{\theta}^T \frac{\partial \lambda_i}{\partial \mathbf{\theta}} \quad (5.4)$$

$$\phi_{Ri} = \phi_{Roi} + \frac{\partial \phi_{Ri}}{\partial \mathbf{\theta}} \Delta \mathbf{\theta} \quad , \quad \phi_{Li} = \phi_{Loi} + \frac{\partial \phi_{Li}}{\partial \mathbf{\theta}} \Delta \mathbf{\theta} \quad (5.5)$$

$\Delta \mathbf{\theta} = (\Delta \theta_1, \Delta \theta_2, \dots, \Delta \theta_p)$ is a vector of changes in the unknown parameters. $\frac{\partial \lambda_i}{\partial \mathbf{\theta}}$ is a vector which represents the derivative of the i^{th} eigenvalue λ_i with respect to each unknown parameter in the vector $\mathbf{\theta}$. $\frac{\partial \phi_{Li}}{\partial \mathbf{\theta}}$ and $\frac{\partial \phi_{Ri}}{\partial \mathbf{\theta}}$ are matrices which denote the derivative of the i^{th} column of the left and right eigenvectors with respect to each unknown parameter in the vector $\mathbf{\theta}$.

There is a substantial literature on eigenvalue and eigenvector derivatives for undamped systems e.g [97-103]. The aim of this chapter is to extend the idea for eigenvalue and eigenvector derivatives for undamped systems to the case of eigenvalue and eigenvector derivatives of general second order system using *structure preserving equivalences*. This approach resolves the completely artificial phenomenon that the eigenvalue and eigenvector derivatives become “undefined” at instants when modification of, say, a damping parameter causes a pair of complex eigenvalues to turn into a pair of real eigenvalues or vice-versa. It also has the advantage of being applicable to cases where any one or more of the system matrices are singular.

In this chapter, section 5.1 is devoted to outlining existing methods for eigenvalue and eigenvector derivatives. Section 5.2 deals with eigenvalue and eigenvector

derivatives for undamped systems. Section 5.3 shows very concisely how (in effect) eigenvalue derivatives may be obtained. Instead of referring to the eigenvalues directly, however, the *diagonalised system* is discussed. Section 5.5 shows concisely how the equivalent of eigenvector derivatives may be obtained. Instead of referring to the eigenvectors explicitly, the *diagonalising transformations* are needed. In section 5.7, the *structure preserving equivalences* (SPEs) are described. Section 5.8 extends the logic of Section 5.3 to finding the derivatives of the *diagonalised system* in the context of damped systems. Section 5.8.1 shows how eigenvalues and their derivatives can be computed from the values and derivatives of the diagonalised system. Section 5.9 extends the logic of Section 5.5 to finding the derivatives of the *diagonalising transformation* in the context of damped systems. Section 5.9.1 shows how the eigenvectors and their derivatives can be extracted from the diagonalising transformation and their derivatives.

The subsequent section contains four examples. In the first example, there is a pair of repeated real roots and the derivatives of two of the eigenvalues and their associated eigenvectors are undefined. This example is generated by adjusting a damping parameter such that at one specific value (corresponding to $\sigma = 0$ where σ is a scalar in vector of parameters θ), higher values result in two distinct real roots and lower values result in a complex conjugate pair of eigenvalues. In the second example, the mass matrix is singular at $\sigma = 0$. The conventional concepts of eigenvalue derivatives encounter difficulty in such cases as the rate of change of infinity is difficult to comprehend. The third example is a case of an undamped system having non-symmetric mass and stiffness matrices where some of the eigenvalues of the system are complex. This example shows that whilst eigenvalue and eigenvector derivatives

can still be “made to work”, the approach to these derivatives through the SPEs is far more elegant and involves only real-valued quantities. The fourth and final example addresses a system of recognisable structure having both singular mass matrix and a pair of identical real roots.

5.1 Eigenvalue and eigenvector derivative methods

There is already a substantial literature on eigenvalue and eigenvector derivatives for undamped systems. Fox and Kapoor [97] provided a method applicable to symmetric undamped systems to calculate $\frac{\partial \lambda_i}{\partial \boldsymbol{\theta}}$ and $\frac{\partial \boldsymbol{\phi}_i}{\partial \boldsymbol{\theta}}$ in equations (5.4) and (5.5) respectively.

The eigenvalue λ_i with corresponding eigenvector $\{\boldsymbol{\phi}_i\}$ satisfy the following equations

$$(\mathbf{K} - \lambda_i \mathbf{M})\boldsymbol{\phi}_i = 0 \quad (5.6)$$

$$\boldsymbol{\phi}_i^T (\mathbf{K} - \lambda_i \mathbf{M}) = 0 \quad (5.7)$$

The derivative of the eigenvalue λ_i can be found from differentiating equation (5.6) with respect to $\boldsymbol{\theta}$ which gives

$$(\mathbf{K} - \lambda_i \mathbf{M})\frac{\partial \boldsymbol{\phi}_i}{\partial \boldsymbol{\theta}} = -\left(\frac{\partial \mathbf{K}}{\partial \boldsymbol{\theta}} - \frac{\partial \lambda_i}{\partial \boldsymbol{\theta}} \mathbf{M} - \lambda_i \frac{\partial \mathbf{M}}{\partial \boldsymbol{\theta}}\right)\boldsymbol{\phi}_i \quad (5.8)$$

and pre-multiplication equation (5.8) by $\boldsymbol{\phi}_i^T$ gives

$$\boldsymbol{\phi}_i^T (\mathbf{K} - \lambda_i \mathbf{M})\frac{\partial \boldsymbol{\phi}_i}{\partial \boldsymbol{\theta}} = -\boldsymbol{\phi}_i^T \left(\frac{\partial \mathbf{K}}{\partial \boldsymbol{\theta}} - \frac{\partial \lambda_i}{\partial \boldsymbol{\theta}} \mathbf{M} - \lambda_i \frac{\partial \mathbf{M}}{\partial \boldsymbol{\theta}}\right)\boldsymbol{\phi}_i \quad (5.9)$$

The left hand side of equation (5.9) vanishes by virtue of equation (5.7). The eigenvalue derivatives can be expressed in the following form

$$\frac{\partial \lambda_i}{\partial \boldsymbol{\theta}} = \boldsymbol{\phi}_i^T \left(\frac{\partial \mathbf{K}}{\partial \boldsymbol{\theta}} - \lambda_i \frac{\partial \mathbf{M}}{\partial \boldsymbol{\theta}}\right)\boldsymbol{\phi}_i / \boldsymbol{\phi}_i^T \mathbf{M} \boldsymbol{\phi}_i \quad (5.10)$$

Equation (2.9) in Chapter 2 expressed the mass normalisation condition that

$$\boldsymbol{\phi}^T \mathbf{M} \boldsymbol{\phi} = 1 \quad (5.11)$$

The eigenvalue derivatives for eigenvector mass normalisation can be written as

$$\frac{\partial \lambda_i}{\partial \boldsymbol{\theta}} = \boldsymbol{\phi}^T \left(\frac{\partial \mathbf{K}}{\partial \boldsymbol{\theta}} - \lambda_i \frac{\partial \mathbf{M}}{\partial \boldsymbol{\theta}} \right) \boldsymbol{\phi} \quad (5.12)$$

Equation (5.12) involves the i^{th} eigenvalues and the corresponding eigenvector of interest. The derivative of the i^{th} eigenvector can be obtained from equation (5.8) which can be rewritten as

$$\begin{aligned} (\mathbf{K} - \lambda_i \mathbf{M}) \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\theta}} &= \left(\left(\boldsymbol{\phi}^T \left(\frac{\partial \mathbf{K}}{\partial \boldsymbol{\theta}} - \lambda_i \frac{\partial \mathbf{M}}{\partial \boldsymbol{\theta}} \right) \boldsymbol{\phi} \right) \mathbf{M} + \lambda_i \frac{\partial \mathbf{M}}{\partial \boldsymbol{\theta}} - \frac{\partial \mathbf{K}}{\partial \boldsymbol{\theta}} \right) \boldsymbol{\phi} \\ &= \mathbf{F}_i \end{aligned} \quad (5.13)$$

Since $(\mathbf{K} - \lambda_i \mathbf{M})$ is a singular matrix of rank $(n-1)$, an extra equation is needed in addition to equation (5.8) to calculate $\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\theta}}$. Differentiating equation (5.11) with respect to $\boldsymbol{\theta}$ yields

$$2\boldsymbol{\phi}^T \mathbf{M} \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\theta}} = -\boldsymbol{\phi}^T \frac{\partial \mathbf{M}}{\partial \boldsymbol{\theta}} \boldsymbol{\phi} \quad (5.14)$$

Together equations (5.8) and (5.14) comprise $(n+1) \times n$ matrix.

$$\begin{bmatrix} (\mathbf{K} - \lambda_i \mathbf{M}) \\ 2\boldsymbol{\phi}^T \mathbf{M} \end{bmatrix} \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\theta}} = - \begin{bmatrix} \frac{\partial (\mathbf{K} - \lambda_i \mathbf{M})}{\partial \boldsymbol{\theta}} \\ \boldsymbol{\phi}^T \frac{\partial \mathbf{M}}{\partial \boldsymbol{\theta}} \end{bmatrix} \boldsymbol{\phi} \quad (5.15)$$

Pre-multiplying equation (5.15) by $\begin{bmatrix} (\mathbf{K} - \lambda_i \mathbf{M}) \\ \boldsymbol{\phi}^T \mathbf{M} \end{bmatrix}^T$ yields

$$\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\theta}} = - \left[(\mathbf{K} - \lambda_i \mathbf{M})^T (\mathbf{K} - \lambda_i \mathbf{M}) + 2\mathbf{M}^T \boldsymbol{\phi} \boldsymbol{\phi}^T \mathbf{M} \right]^{-1} \left[(\mathbf{K} - \lambda_i \mathbf{M})^T \frac{\partial (\mathbf{K} - \lambda_i \mathbf{M})}{\partial \boldsymbol{\theta}} + \mathbf{M}^T \boldsymbol{\phi} \boldsymbol{\phi}^T \frac{\partial \mathbf{M}}{\partial \boldsymbol{\theta}} \right] \boldsymbol{\phi} \quad (5.16)$$

Fox and Kapoor [97] proposed another expression to solve equation (5.8). The derivative of the i^{th} eigenvector can be expressed as a linear combination of the complete set of vectors of Φ .

$$\frac{\partial \Phi_i}{\partial \theta} = \sum_{k=1}^n c_k \Phi_k = \Phi \mathbf{c} \quad (5.17)$$

Substitute equation (5.17) into equation (5.8) and pre-multiply by Φ_k^T to calculate c_k

$$\Phi_k^T (\mathbf{K} - \lambda_i \mathbf{M}) c_k \Phi_k = -\Phi_k^T \left(\frac{\partial \mathbf{K}}{\partial \theta} - \frac{\partial \lambda_i}{\partial \theta} \mathbf{M} - \lambda_i \frac{\partial \mathbf{M}}{\partial \theta} \right) \Phi_i \quad (5.18)$$

The general scalar c_k can be written in the form

$$c_k = -\Phi_k^T \left(\frac{\partial \mathbf{K}}{\partial \theta} - \frac{\partial \lambda_i}{\partial \theta} \mathbf{M} - \lambda_i \frac{\partial \mathbf{M}}{\partial \theta} \right) \Phi_i / \Phi_k^T (\mathbf{K} - \lambda_i \mathbf{M}) \Phi_k \quad k \neq i \quad (5.19)$$

For eigenvectors mass normalised, the c_k can be rewritten

$$c_k = \Phi_k^T \left(\frac{\partial \mathbf{K}}{\partial \theta} - \frac{\partial \lambda_i}{\partial \theta} \mathbf{M} - \lambda_i \frac{\partial \mathbf{M}}{\partial \theta} \right) \Phi_i / (\lambda_i - \lambda_k) \quad k \neq i \quad (5.20)$$

To find c_i we apply the mass normalisation condition equation (5.14)

$$c_i = -\Phi_i^T \frac{\partial \mathbf{M}}{\partial \theta} \Phi_i / 2 \quad k = i \quad (5.21)$$

Calculating the coefficients in equation (5.17) using equations (5.20) and (5.21) requires the complete set of eigenvectors but does not involve inverting any matrix.

These expressions have been developed by numerous authors [98-103] to determine the eigenvalue and eigenvector derivatives for more general non-symmetric undamped systems. For large eigenvalue problems, the calculation of eigenvalue derivatives is simple and straightforward, whereas the calculation of eigenvector derivatives is more complicated.

Nelson [104] simplified the procedure for calculating eigenvector derivatives of symmetric and non-symmetric undamped systems so that only the eigenvalue and eigenvector under consideration are required. The procedure of this method is to modify the eigenvalue problem matrix $(\mathbf{K} - \lambda_i \mathbf{M})$ of rank $(n-1)$ in equation (5.8) to matrix of rank n . The method shows that right and left eigenvector derivatives $\left\{ \frac{\partial \phi_{Ri}}{\partial \theta}, \frac{\partial \phi_{Li}}{\partial \theta} \right\}$ have a unique expression in terms of $(n-1)$ eigenvectors of the system, except the i^{th} eigenvector

$$\frac{\partial \phi_{Ri}}{\partial \theta} = \sum_{k=1, k \neq i}^n c_k \phi_{Rk} + c_{Ri} \phi_{Ri} = \mathbf{v}_{Ri} + c_{Ri} \phi_{Ri} \quad (5.22)$$

$$\frac{\partial \phi_{Li}}{\partial \theta} = \sum_{k=1, k \neq i}^n c_k \phi_{Lk} + c_{Li} \phi_{Li} = \mathbf{v}_{Li} + c_{Li} \phi_{Li} \quad (5.23)$$

$\{\mathbf{v}_{Ri}, \mathbf{v}_{Li}\}$ are vectors related to the right and left eigenvectors. The solution of the vectors $\{\mathbf{v}_{Ri}, \mathbf{v}_{Li}\}$ in equations (5.22) and (5.23) can be computed by partitioning the matrix on the left hand side of equation (5.13). This partition depends on the location of the k^{th} pivotal components $\{\mathbf{v}_{Rk}, \mathbf{v}_{Lk}\}$ of the vectors $\{\mathbf{v}_{Ri}, \mathbf{v}_{Li}\}$. The k^{th} location is where the inner product of ϕ_{Rk} and ϕ_{Lk} is maximum - $\left\{ |\phi_{Rk}| \bullet |\phi_{Lk}| \text{ is maximum} \right\}$. The complete solution for right and left eigenvector derivatives $\left\{ \frac{\partial \phi_{Ri}}{\partial \theta}, \frac{\partial \phi_{Li}}{\partial \theta} \right\}$ is

$$\frac{\partial \phi_{Ri}}{\partial \theta} = \begin{Bmatrix} \mathbf{v}_{R1} \\ \vdots \\ 0 \\ \vdots \\ \mathbf{v}_{Rn} \end{Bmatrix} + c_{Ri} \begin{Bmatrix} \phi_{R1} \\ \vdots \\ \phi_{Rk} \\ \vdots \\ \phi_{Rn} \end{Bmatrix} \quad (5.24)$$

$$\frac{\partial \phi_{Li}}{\partial \theta} = \begin{Bmatrix} \mathbf{v}_{L1} \\ \vdots \\ 0 \\ \vdots \\ \mathbf{v}_{Ln} \end{Bmatrix} + c_{Li} \begin{Bmatrix} \phi_{L1} \\ \vdots \\ \phi_{Lk} \\ \vdots \\ \phi_{Ln} \end{Bmatrix} \quad (5.25)$$

The superscript “ \wedge ” is used to indicate the n entries of the vectors \mathbf{v}_{Ri} and ϕ_{Ri} and the k^{th} pivotal components $\{\mathbf{v}_{Rk}, \mathbf{v}_{Lk}\}$ of vectors $\{\mathbf{v}_{Ri}, \mathbf{v}_{Li}\}$ are zero. The entries in the k^{th} row and column of the eigenvalue problem matrix in equation (5.13) are zero and the location (k, k) of that matrix is 1.

$$\begin{bmatrix} (\mathbf{K}_{11} - \lambda_i \mathbf{M}_{11}) & \cdots & 0 & \cdots & (\mathbf{K}_{1n} - \lambda_i \mathbf{M}_{1n}) \\ \cdots & \cdots & 0 & \cdots & \cdots \\ 0 & 0 & 1 & 0 & 0 \\ \cdots & \cdots & 0 & \cdots & \cdots \\ (\mathbf{K}_{n1} - \lambda_i \mathbf{M}_{n1}) & \cdots & 0 & \cdots & (\mathbf{K}_{nn} - \lambda_i \mathbf{M}_{nn}) \end{bmatrix} \begin{Bmatrix} \mathbf{v}_{R1} \\ \vdots \\ \mathbf{v}_{Rk} \\ \vdots \\ \mathbf{v}_{Rn} \end{Bmatrix} = \begin{Bmatrix} \hat{\mathbf{f}}_{R1} \\ \vdots \\ 0 \\ \vdots \\ \hat{\mathbf{f}}_{Rn} \end{Bmatrix} \quad (5.26)$$

The vector \mathbf{v}_{Ri} can be solved directly from equation (5.26) since the matrix on the left hand side of equation (5.26) is of rank n . The c_{Ri} is calculated using the norm for ϕ_{Ri} that can be put in the form

$$\phi_{Ri}^T \mathbf{M} \phi_{Ri} = 1 \quad (5.27)$$

Differentiating equation (5.27) with respect to θ yields

$$2\text{Re} \left(\phi_{Ri}^H \mathbf{M} \frac{\partial \phi_{Ri}}{\partial \theta} \right) + \phi_{Ri}^H \frac{\partial \mathbf{M}}{\partial \theta} \phi_{Ri} = 0 \quad (5.28)$$

Re is the real part of a complex number. Substitute equation (5.22) into equation (5.28) to calculate c_{Ri} yields

$$c_{Ri} = - \left(\text{Re} \left(\phi_{Ri}^H \mathbf{M} \mathbf{v}_{Ri} \right) + 0.5 \phi_{Ri}^H \frac{\partial \mathbf{M}}{\partial \theta} \phi_{Ri} \right) \quad (5.29)$$

The left eigenvector derivatives $\frac{\partial \phi_{Li}}{\partial \theta}$ can be found by differentiating equation (2.4)

with respect to θ and then following the same procedure.

Most of the existing methods explored eigenvalue and eigenvector derivatives of undamped eigenvalue problems. These methods cannot be applied directly for repeated eigenvalues. Many methods [114-118] are proposed for calculating eigenvector derivatives with repeated eigenvalues. All methods suggested by references [114-118] have difficulties to define the exact eigenvector derivatives. Baisheng Wu et al. [119] presented a method for computing eigenvector derivatives with repeated eigenvalues for real symmetric eigensystems. The solution is required to be mass orthogonal with respect to repeated modes. The coefficient matrices are non-singular.

In this chapter, existing methods have been developed and generalised to determine eigenvalue and eigenvector derivatives for undamped systems. The solution required knowledge of the eigenvalues and eigenvectors under consideration. Real diagonalised system, diagonalising transformations and their derivatives have been used in order to calculate the eigenvalue and eigenvector derivatives. These methods overcome the difficulty that eigenvalue derivatives become undefined. The ideas for undamped system are extended for generally damped systems to the concept of structure preserving equivalences. For the purpose of this chapter, differentiation is chosen to be with respect to σ , where σ is a singular scalar parameter of the uncertain parameters vector θ .

5.2 Eigenvalue and eigenvector derivatives for undamped systems

This section provides an approach based on the above existing methods but takes a slightly more general view of the concept of eigenvalues based on the *homogeneous coordinates* approach of [120]. The purpose of this section is to provide a framework for undamped systems which can be extended naturally to damped systems. Aside from this purpose, the only added value in this section is that it caters naturally for the case where the mass matrix may be singular whereas the aforementioned methods do not.

Non-symmetric undamped systems $\{\mathbf{K}, \mathbf{M}\}$ are considered in this section. The stiffness matrix is \mathbf{K} and the mass matrix is \mathbf{M} . It is possible that such systems can produce complex eigenvalues and associated complex eigenvectors. The treatment of this section can be applied to such cases but there may be issues of numerical stability in such cases. The intention for the methods of this section is that they would only ever be applied to the case of real eigenvalues. Where an undamped system has complex eigenvalues, the natural course is to treat it as we propose that all damped systems should be treated (sections 5.8 and 5.9).

In place of the concept of matrices of left and right eigenvectors, we consider transformation matrices $\{\mathbf{T}_L, \mathbf{T}_R\}$ which diagonalise the system according to the following equation

$$\mathbf{T}_L^T \mathbf{K} \mathbf{T}_R = \mathbf{K}_D \quad (5.30)$$

$$\mathbf{T}_L^T \mathbf{M} \mathbf{T}_R = \mathbf{M}_D \quad (5.31)$$

$$\mathbf{T}_L^T \mathbf{D} \mathbf{T}_R = \mathbf{D}_D \quad (5.32)$$

In this section, the eigenvalue and eigenvector derivatives are calculated based on the diagonalised system and diagonalising transformations using equations (5.30) and (5.31) which describe the undamped second order problem $\det(\mathbf{K} - \lambda^2 \mathbf{M}) = 0$. Abuazoum and Garvey [121] considered the eigenvalue and eigenvector derivatives of the first order problem $\det(\mathbf{K} + \lambda \mathbf{D}) = 0$ and the method is completely analogous.

The eigenvalues of this system represent squares of natural frequencies in rad/s. The following general definitions will be used

$$(\mathbf{K} \mathbf{m}_i - \mathbf{M} \mathbf{k}_i) \boldsymbol{\phi}_{Ri} = 0 \quad (5.33)$$

$$\boldsymbol{\phi}_{Li}^T (\mathbf{K} \mathbf{m}_i - \mathbf{M} \mathbf{k}_i) = 0 \quad (5.34)$$

where $\{m_i, k_i\}$ together represent the eigenvalue $\lambda := (k_i / m_i)$ and $\{\boldsymbol{\phi}_{Li}, \boldsymbol{\phi}_{Ri}\}$ represent the corresponding left and right eigenvectors, respectively. For consistency with later sections, we place all emphasis on the pair $\{m_i, k_i\}$ and no emphasis on the eigenvalue itself. In most instances, we could insist that $m_i = 1$. This corresponds to the familiar “mass-normalisation”. However a more generally-applicable constraint is that

$$k_i^2 + m_i^2 = 1 \quad \text{for all } i \quad (5.35)$$

The definition of the eigenvalues in equations (5.33) and (5.34) can be referred to as *homogeneous coordinates* [120] and the same arbitrary normalisation of each pair $\{m_i, k_i\}$ is used there. This formulation normalisation admits the possibility that either $m_i = 0$ or $k_i = 0$ and it is possible to express $\{m_i, k_i\}$, respectively, as the cosine and sine of a single scalar angle but there is no particular value in pursuing this expression here.

The following orthogonality relationships are easily proven

$$\boldsymbol{\phi}_{Li}^T \mathbf{K} \boldsymbol{\phi}_{Rj} = 0 \quad \forall \quad m_i k_j \neq k_i m_j \quad (5.36)$$

$$\boldsymbol{\phi}_{Li}^T \mathbf{M} \boldsymbol{\phi}_{Rj} = 0 \quad \forall \quad m_i k_j \neq k_i m_j. \quad (5.37)$$

In this section, we will consider that the eigenvalues are all distinct. In later sections dealing with damped systems, this condition will be relaxed to the milder condition that no pairs of eigenvalues are repeated. Collecting the left and right eigenvectors in the same order produces

$$\boldsymbol{\Phi}_R = [\boldsymbol{\phi}_{R1} \quad \boldsymbol{\phi}_{R2} \quad \dots \quad \boldsymbol{\phi}_{Rn}] \quad (5.38)$$

$$\boldsymbol{\Phi}_L = [\boldsymbol{\phi}_{L1} \quad \boldsymbol{\phi}_{L2} \quad \dots \quad \boldsymbol{\phi}_{Ln}] \quad (5.39)$$

and both matrices, $\{\boldsymbol{\Phi}_L, \boldsymbol{\Phi}_R\}$, will be invertible since the eigenvalues are distinct. Then diagonal matrices, $\{\mathbf{K}_D, \mathbf{M}_D\}$ are related to the original stiffness and mass matrices $\{\mathbf{K}_o, \mathbf{M}_o\}$ through the conventional equivalences

$$\mathbf{K}_D = \boldsymbol{\Phi}_L^T \mathbf{K} \boldsymbol{\Phi}_R \quad (5.40)$$

$$\mathbf{M}_D = \boldsymbol{\Phi}_L^T \mathbf{M} \boldsymbol{\Phi}_R \quad (5.41)$$

Note that the normalization of equation (5.35) controls the scaling of $(\boldsymbol{\Phi}_L^T \mathbf{K} \boldsymbol{\Phi}_R)$ and $(\boldsymbol{\Phi}_L^T \mathbf{M} \boldsymbol{\Phi}_R)$. The i^{th} diagonal entries of $\{\mathbf{K}_D, \mathbf{M}_D\}$ are $\{k_i, m_i\}$, respectively. We see that the eigenvectors define a *diagonalising transformation* which maps the original system matrices $\{\mathbf{K}, \mathbf{M}\}$ onto the diagonal matrices $\{\mathbf{K}_D, \mathbf{M}_D\}$.

5.3 Rates of change of the diagonalised matrices for undamped systems

Differentiate equations (5.40) and (5.41) with respect to some scalar parameter, σ . The dot notation employed above the matrices here and henceforth indicates a derivative with respect to σ .

$$\dot{\mathbf{K}}_D = \dot{\Phi}_L^T \mathbf{K} \Phi_R + \Phi_L^T \dot{\mathbf{K}} \Phi_R + \Phi_L^T \mathbf{K} \dot{\Phi}_R \quad (5.42)$$

$$\dot{\mathbf{M}}_D = \dot{\Phi}_L^T \mathbf{M} \Phi_R + \Phi_L^T \dot{\mathbf{M}} \Phi_R + \Phi_L^T \mathbf{M} \dot{\Phi}_R \quad (5.43)$$

The derivatives of each pair, $\{k_i, m_i\}$, must be determined in isolation. Throughout this paper, we use the notation, \mathbf{e}_i , to denote the i^{th} column of the $(n \times n)$ identity matrix and we will denote by \mathbf{S}_i the matrix containing the remaining $(n-1)$ columns. Pre-multiplying equations (5.42) and (5.43) by \mathbf{e}_i^T and post-multiplying them by \mathbf{e}_i yields

$$\dot{k}_i = \dot{\Phi}_{Li}^T \mathbf{K} \Phi_{Ri} + \Phi_{Li}^T \dot{\mathbf{K}} \Phi_{Ri} + \Phi_{Li}^T \mathbf{K} \dot{\Phi}_{Ri} \quad (5.44)$$

$$\dot{m}_i = \dot{\Phi}_{Li}^T \mathbf{M} \Phi_{Ri} + \Phi_{Li}^T \dot{\mathbf{M}} \Phi_{Ri} + \Phi_{Li}^T \mathbf{M} \dot{\Phi}_{Ri} \quad (5.45)$$

Multiplying equation (5.44) by m_i , multiplying equation (5.45) by k_i and subtracting the latter result from the former gives

$$m_i \dot{k}_i - k_i \dot{m}_i = \dot{\Phi}_{Li}^T (m_i \mathbf{K}_1 - k_i \mathbf{M}_1) \Phi_{Ri} \quad (5.46)$$

Equations (5.36) and (5.37) were invoked to cancel terms from equation (5.46) and equations (5.1) and (5.3) were also applied to replace $\{\dot{\mathbf{K}}, \dot{\mathbf{M}}\}$ by $\{\mathbf{K}_1, \mathbf{M}_1\}$, respectively. Evidently, equation (5.46) is insufficient to determine $\{\dot{k}_i, \dot{m}_i\}$ uniquely. The normalisation information of equation (5.35) provides the necessary second equation

$$k_i \dot{k}_i + m_i \dot{m}_i = 0 \quad (5.47)$$

Combining equations (5.46) and (5.47) yields

$$\begin{bmatrix} m_i & -k_i \\ k_i & m_i \end{bmatrix} \begin{bmatrix} \dot{k}_i \\ \dot{m}_i \end{bmatrix} = \begin{bmatrix} \diamond \\ 0 \end{bmatrix} \quad (5.48)$$

where the diamond symbol, \diamond , is used here (and later) to represent a known scalar quantity. In the present case, it happens to be $(\phi_{Li}^T (m_i \mathbf{K}_1 - k_i \mathbf{M}_1) \phi_{Ri})$. Note that the (2×2) matrix of Eq. (5.48) is an orthogonal matrix.

5.4 Orthogonal transformation matrix - Householder Reflection

Any orthogonal transformation in the plane or three dimensional space is a rotation. It is possibly combined with a reflection in a straight line or a plane [40]. Chandrupatla and Conrads [122] presented a technical note about the use of Householder reflection in place of Givens rotation in matrix decomposition. The Householder reflection is used to decompose a fully populated matrix into bidiagonal form. The Householder matrix is defined as

$$\mathbf{Q} = (\mathbf{I} - 2\mathbf{u}\mathbf{u}^T) \quad (5.49)$$

where \mathbf{u} is a unit vector defining the direction of a reflection. In two dimensions, if

$\mathbf{u} = [\mathbf{u}_i \quad \mathbf{u}_j]^T$ and for given a matrix Φ ,

$$\Phi = [\phi_i \quad \dots \quad \phi_k] = \begin{bmatrix} \phi_{ii} & \dots & \phi_{ik} \\ \phi_{ji} & \dots & \phi_{jk} \end{bmatrix} \quad (5.50)$$

The matrix \mathbf{Q} can be written as

$$\mathbf{Q}_{i,j} = \begin{bmatrix} 1 - 2\mathbf{u}_i^2 & -2\mathbf{u}_i\mathbf{u}_j \\ -2\mathbf{u}_i\mathbf{u}_j & 1 - 2\mathbf{u}_j^2 \end{bmatrix} \quad (5.51)$$

Since the vector \mathbf{u} is a unit vector, $\mathbf{u}_i^2 + \mathbf{u}_j^2 = 1$ and $[1 - 2\mathbf{u}_i]^2 + [-2\mathbf{u}_i\mathbf{u}_j]^2 = 1$. Then, the matrix $\mathbf{Q}_{i,j}$ is proposed to be

$$\mathbf{Q}_{i,j} = \begin{bmatrix} c & s \\ s & -c \end{bmatrix} \quad (5.52)$$

Where $c^2 + s^2 = 1$, the c and s are chosen such that

$$\mathbf{Q}_{i,j}\boldsymbol{\phi} = \begin{bmatrix} c & s \\ s & -c \end{bmatrix} \begin{Bmatrix} \phi_i \\ \phi_{ji} \end{Bmatrix} = \begin{Bmatrix} \|\boldsymbol{\phi}\| \\ 0 \end{Bmatrix} \quad (5.53)$$

Therefore, $c = \frac{\phi_i}{\|\boldsymbol{\phi}\|}$, $s = \frac{\phi_{ji}}{\|\boldsymbol{\phi}\|}$, $\|\boldsymbol{\phi}\| = \sqrt{\phi_i^2 + \phi_{ji}^2}$. The symbol $\|\cdot\|$ denotes the Euclidean norm.

$$\mathbf{Q}_{i,j}\boldsymbol{\phi}_k = \begin{bmatrix} c & s \\ s & -c \end{bmatrix} \begin{Bmatrix} \phi_k \\ \phi_{jk} \end{Bmatrix} = \begin{bmatrix} c\phi_k + s\phi_{jk} \\ s\phi_k - c\phi_{jk} \end{bmatrix}_{k=1\dots n, k \neq i} \quad (5.54)$$

5.5 Rates of change of the diagonalising transformation for undamped system

Applying equations (5.33) and (5.34) for all of the eigenvectors simultaneously results in

$$(\mathbf{K}\boldsymbol{\Phi}_R\mathbf{M}_D - \mathbf{M}\boldsymbol{\Phi}_R\mathbf{K}_D) = 0 \quad (5.55)$$

$$(\mathbf{M}_D\boldsymbol{\Phi}_L^T\mathbf{K} - \mathbf{K}_D\boldsymbol{\Phi}_L^T\mathbf{M}) = 0 \quad (5.56)$$

Differentiating equation (5.55) with respect to σ yields

$$\mathbf{K}\dot{\boldsymbol{\Phi}}_R\mathbf{M}_D - \mathbf{M}\dot{\boldsymbol{\Phi}}_R\mathbf{K}_D = (\mathbf{M}\boldsymbol{\Phi}_R\dot{\mathbf{K}}_D + \mathbf{M}_1\boldsymbol{\Phi}_R\mathbf{K}_D) - (\mathbf{K}_1\boldsymbol{\Phi}_R\mathbf{M}_D + \mathbf{K}\boldsymbol{\Phi}_R\dot{\mathbf{M}}_D) \quad (5.57)$$

The quantities on the right hand side of equation (5.57) are known. Note that because $\{\mathbf{K}_D, \mathbf{M}_D\}$ are both diagonal, column i of the left hand side involves only the eigenvector, ϕ_{Ri} . Thus for the general i^{th} right eigenvector,

$$(m_i \mathbf{K} - k_i \mathbf{M}) \dot{\boldsymbol{\phi}}_{Ri} = \left[(\mathbf{M} \dot{k}_i + \mathbf{M}_1 k_i) - (\mathbf{K}_1 m_i + \mathbf{K} \dot{m}_i) \right] \boldsymbol{\phi}_{Ri} \quad (5.58)$$

Similarly for the left eigenvector, by differentiating equation (5.56) and taking row i we can obtain

$$\dot{\boldsymbol{\phi}}_{Li}^T (m_i \mathbf{K} - k_i \mathbf{M}) = \boldsymbol{\phi}_{Li}^T \left[(\mathbf{M} \dot{k}_i + \mathbf{M}_1 k_i) - (\mathbf{K}_1 m_i + \mathbf{K} \dot{m}_i) \right] \quad (5.59)$$

Now, $(m_i \mathbf{K} - k_i \mathbf{M})$ has one zero singular value (recall our assumption of no repeated eigenvalues) and hence the solution of equations (5.58) and (5.59) is not simply a matter of finding the inverse of this matrix. In fact, these equations each reveal the fundamental truth that it is only possible to know $(n-1)$ independent facts about the rate of change of any one eigenvector (either left or right). The reason for the remaining unknown is that even when the system is not changing, any multiple of $\boldsymbol{\phi}_{Ri}$ can be added onto $\boldsymbol{\phi}_{Ri}$ itself without compromising its legitimacy as a right eigenvector. The same is true for $\boldsymbol{\phi}_{Li}$. Nelson [104] showed that by writing the eigenvector derivative as a linear combination of the eigenvectors, it becomes clear what can be known about the eigenvector derivative and what cannot. We adopt a more direct (and more efficient) approach here.

Begin by calculating orthogonal matrices, $\{\mathbf{Q}_{Li}, \mathbf{Q}_{Ri}\}$ such that

$$\mathbf{Q}_{Li} \boldsymbol{\phi}_{Li} = \mathbf{e}_i \alpha_{Li} \quad (5.60)$$

$$\mathbf{Q}_{Ri} \boldsymbol{\phi}_{Ri} = \mathbf{e}_i \alpha_{Ri} \quad (5.61)$$

where $\{\alpha_{Li}, \alpha_{Ri}\}$ are arbitrary real scalars. Such matrices $\{\mathbf{Q}_{Li}, \mathbf{Q}_{Ri}\}$ are easily achieved as Householder reflections [123] which, in addition to being orthogonal, have the very attractive properties of being symmetric and low-rank modifications of the

identity. Recalling that \mathbf{S}_i represents the $(n \times n)$ identity matrix from which the i^{th} column has been removed, it is clear that

$$\left(\mathbf{S}_i^T \mathbf{Q}_{Li} \boldsymbol{\phi}_{Li} = 0 = \mathbf{S}_i^T \mathbf{Q}_{Ri} \boldsymbol{\phi}_{Ri} \right) \quad (5.62)$$

Now, write the two desired vector derivatives as

$$\dot{\boldsymbol{\phi}}_{Li} = \left(\mathbf{Q}_{Li}^T \mathbf{S}_i \right) \mathbf{g}_{Li} + \beta_{Li} \boldsymbol{\phi}_{Li} \quad (5.63)$$

$$\dot{\boldsymbol{\phi}}_{Ri} = \left(\mathbf{Q}_{Ri}^T \mathbf{S}_i \right) \mathbf{g}_{Ri} + \beta_{Ri} \boldsymbol{\phi}_{Ri} \quad (5.64)$$

The vectors $\{\mathbf{g}_{Li}, \mathbf{g}_{Ri}\}$ each have $(n-1)$ entries and these can be computed directly from equations (5.58) and (5.59), respectively. The transpose symbols in equations (5.63) and (5.64) are not necessary if $\{\mathbf{Q}_{Li}, \mathbf{Q}_{Ri}\}$ have been calculated as Householder reflections but we allow that $\{\mathbf{Q}_{Li}, \mathbf{Q}_{Ri}\}$ are not necessarily Householder reflections. Many other options are also available.

The Householder reflection matrices have the form as in equation (5.49)

$$\mathbf{Q}_{Li} = \left(\mathbf{I} - 2\mathbf{u}_L \mathbf{u}_L^T \right) \quad , \quad \mathbf{Q}_{Ri} = \left(\mathbf{I} - 2\mathbf{u}_R \mathbf{u}_R^T \right) \quad (5.65)$$

Vectors $\{\mathbf{u}_L, \mathbf{u}_R\}$ are unit vectors where $\|\mathbf{u}_L\| = \sqrt{\mathbf{u}_L^T \mathbf{u}_L} = 1$ and $\|\mathbf{u}_R\| = \sqrt{\mathbf{u}_R^T \mathbf{u}_R} = 1$. These vectors can be calculated from eigenvectors $\{\boldsymbol{\phi}_{Li}, \boldsymbol{\phi}_{Ri}\}$ as

$$\begin{aligned} \mathbf{u}_L &= \boldsymbol{\phi}_{Li} - \mathbf{e}_i \alpha_L \quad , \quad \mathbf{u}_L = \frac{\mathbf{u}_L}{\|\mathbf{u}_L\|} \\ \mathbf{u}_R &= \boldsymbol{\phi}_{Ri} - \mathbf{e}_i \alpha_R \quad , \quad \mathbf{u}_R = \frac{\mathbf{u}_R}{\|\mathbf{u}_R\|} \end{aligned} \quad (5.66)$$

where $\{\alpha_L, \alpha_R\}$ can be calculated from the following expression

$$\alpha_{Li} = \frac{\|\boldsymbol{\phi}_{Li}\|}{\|\mathbf{e}_i\|} \quad , \quad \alpha_{Ri} = \frac{\|\boldsymbol{\phi}_{Ri}\|}{\|\mathbf{e}_i\|} \quad (5.67)$$

To obtain \mathbf{g}_{Ri} , substitute for $\dot{\phi}_{Ri}$ in equation (5.58). The term involving the unknown scalar, β_{Ri} , vanishes naturally and the result is a set of n consistent equations with $(n-1)$ unknowns. The most stable solution of these equations is achieved through use of the left pseudo-inverse. To obtain \mathbf{g}_{Li} , substitute for $\dot{\phi}_{Li}$ in equation (5.59). The term involving the unknown scalar, β_{Li} , vanishes naturally and the result is a set of n consistent equations with $(n-1)$ unknowns. The most stable solution of these equations is achieved through use of the right pseudo-inverse.

Having evaluated vectors $\{\mathbf{g}_{Li}, \mathbf{g}_{Ri}\}$, it remains only to set values for the unknown scalars $\{\beta_{Li}, \beta_{Ri}\}$. The normalisation of equation (5.35) provides one equation governing these since $\phi_{Li}^T \mathbf{K} \phi_{Ri} = k_i$ and $\phi_{Li}^T \mathbf{M} \phi_{Ri} = m_i$. Thus

$$k_i \left(\dot{\phi}_{Li}^T \mathbf{K} \phi_{Ri} + \phi_{Li}^T \mathbf{K} \dot{\phi}_{Ri} + \phi_{Li}^T \mathbf{K} \dot{\phi}_{Ri} \right) + m_i \left(\dot{\phi}_{Li}^T \mathbf{M} \phi_{Ri} + \phi_{Li}^T \mathbf{M} \dot{\phi}_{Ri} + \phi_{Li}^T \mathbf{M} \dot{\phi}_{Ri} \right) = 0 \quad (5.68)$$

Substituting for $\{\dot{\phi}_{Li}, \dot{\phi}_{Ri}\}$ using equations (5.63) and (5.64) transforms equation (5.68) into a linear equation in $\{\beta_{Li}, \beta_{Ri}\}$. Obviously, one linear equation is not sufficient to determine two unknowns and a further arbitrary decision must be made. For symmetric undamped systems, the left and right eigenvectors can be forced to be identical and in this case, $\beta_{Li} = \beta_{Ri}$. In more general cases, a reasonable strategy is to maintain the following condition

$$\phi_{Li}^T \phi_{Li} = \phi_{Ri}^T \phi_{Ri} \quad \text{for all } i \quad (5.69)$$

and this clearly leads to a second linear equation in $\{\beta_{Li}, \beta_{Ri}\}$. The problem of determining the eigenvector derivatives is now solved. Moreover, notice that in order

to determine the derivatives of the i^{th} left and right eigenvectors, it is not necessary to know any other eigenvectors.

5.6 Eigenvalues and eigenvectors for damped systems

The governing equation of motion of a general second order damped system is

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f} \quad (5.70)$$

where $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ are the system matrices which satisfy the following equations

$$(\lambda_i^2 \mathbf{M} + \lambda_i \mathbf{D} + \mathbf{K}) \phi_{Ri} = 0 \quad (5.71)$$

$$\phi_{Li}^T (\lambda_i^2 \mathbf{M} + \lambda_i \mathbf{D} + \mathbf{K}) = 0 \quad (5.72)$$

Defining $\{\phi_{Li}, \phi_{Ri}\}$ as the i^{th} left and right eigenvectors, where $i = \{1, 2, \dots, n\}$

In state space, the $(2n \times 2n)$ left and right eigenvector matrices $\{\underline{\Phi}_L, \underline{\Phi}_R\}$ diagonalise the Lancaster augmented matrices (LAM's) $\{\underline{\mathbf{K}}, \underline{\mathbf{D}}, \underline{\mathbf{M}}\}$. The eigenvalues $\underline{\Lambda}$ might be complex or real or mixed (real and complex). If all eigenvalues are complex $\underline{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_1^*, \lambda_2^*, \dots, \lambda_n^*)$

$$\begin{bmatrix} \Phi_L & \Phi_L^* \\ \Lambda \Phi_L & \Lambda^* \Phi_L^* \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \Phi_R & \Phi_R^* \\ \Lambda \Phi_R & \Lambda^* \Phi_R^* \end{bmatrix} = \begin{bmatrix} \Lambda \Psi & \mathbf{0} \\ \mathbf{0} & \Lambda^* \Psi^* \end{bmatrix} = \begin{bmatrix} \Omega & \mathbf{0} \\ \mathbf{0} & \Omega^* \end{bmatrix} \quad (5.73)$$

$$\begin{bmatrix} \Phi_L & \Phi_L^* \\ \Lambda \Phi_L & \Lambda^* \Phi_L^* \end{bmatrix}^T \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \Phi_R & \Phi_R^* \\ \Lambda \Phi_R & \Lambda^* \Phi_R^* \end{bmatrix} = \begin{bmatrix} \Psi & \mathbf{0} \\ \mathbf{0} & \Psi^* \end{bmatrix} \quad (5.74)$$

$$\begin{bmatrix} \Phi_L & \Phi_L^* \\ \Lambda \Phi_L & \Lambda^* \Phi_L^* \end{bmatrix}^T \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Phi_R & \Phi_R^* \\ \Lambda \Phi_R & \Lambda^* \Phi_R^* \end{bmatrix} = \begin{bmatrix} \Psi/\Lambda & \mathbf{0} \\ \mathbf{0} & \Psi^*/\Lambda^* \end{bmatrix} = \begin{bmatrix} \Gamma & \mathbf{0} \\ \mathbf{0} & \Gamma^* \end{bmatrix} \quad (5.75)$$

Equations (5.73), (5.74) and (5.75) can be written in general form as

$$\underline{\Phi}_L^T \underline{\mathbf{M}} \underline{\Phi}_R = \underline{\Omega} \quad (5.76)$$

$$\underline{\Phi}_L^T \underline{\mathbf{D}} \underline{\Phi}_R = \underline{\Psi} \quad (5.77)$$

$$\underline{\Phi}_L^T \underline{\mathbf{K}} \underline{\Phi}_R = \underline{\Gamma} \quad (5.78)$$

where $\{\underline{\Omega}, \underline{\Psi}, \underline{\Gamma}\}$ are diagonal matrices $\underline{\Omega} = \text{diag}(\omega_1, \omega_2, \dots, \omega_n)$, $\underline{\Psi} = \text{diag}(\psi_1, \psi_2, \dots, \psi_n)$ and $\underline{\Gamma} = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_n)$. If the left and right matrices of complete eigenvectors $\{\underline{\Phi}_L, \underline{\Phi}_R\}$ are normalised using equation (5.74) and setting $\underline{\Psi} = \underline{\mathbf{I}}$, the values of $\{\underline{\Omega}, \underline{\Psi}, \underline{\Gamma}\}$ are still diagonal and can be expressed as $\{\underline{\Lambda}, \underline{\mathbf{I}}, \underline{\Lambda}^{-1}\}$ respectively.

We will select two columns of eigenvectors at a time: the i^{th} column and the complex conjugate of the i^{th} column. Equations (5.73), (5.74) and (5.75) can be written as

$$\begin{bmatrix} \phi_{Li} & \phi_{Li}^* \\ \lambda_i \phi_{Li} & \lambda_i^* \phi_{Li}^* \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \phi_{Ri} & \phi_{Ri}^* \\ \lambda_i \phi_{Ri} & \lambda_i^* \phi_{Ri}^* \end{bmatrix} = \begin{bmatrix} \lambda_i \psi_i & \mathbf{0} \\ \mathbf{0} & \lambda_i^* \psi_i^* \end{bmatrix} = \begin{bmatrix} \omega_i & \mathbf{0} \\ \mathbf{0} & \omega_i^* \end{bmatrix} \quad (5.79)$$

$$\begin{bmatrix} \phi_{Li} & \phi_{Li}^* \\ \lambda_i \phi_{Li} & \lambda_i^* \phi_{Li}^* \end{bmatrix}^T \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \phi_{Ri} & \phi_{Ri}^* \\ \lambda_i \phi_{Ri} & \lambda_i^* \phi_{Ri}^* \end{bmatrix} = \begin{bmatrix} \psi_i & \mathbf{0} \\ \mathbf{0} & \psi_i^* \end{bmatrix} \quad (5.80)$$

$$\begin{bmatrix} \phi_{Li} & \phi_{Li}^* \\ \lambda_i \phi_{Li} & \lambda_i^* \phi_{Li}^* \end{bmatrix}^T \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \phi_{Ri} & \phi_{Ri}^* \\ \lambda_i \phi_{Ri} & \lambda_i^* \phi_{Ri}^* \end{bmatrix} = \begin{bmatrix} \psi_i / \lambda_i & \mathbf{0} \\ \mathbf{0} & \psi_i^* / \lambda_i^* \end{bmatrix} = \begin{bmatrix} \gamma_i & \mathbf{0} \\ \mathbf{0} & \gamma_i^* \end{bmatrix} \quad (5.81)$$

$$\underline{\phi}_{Li}^T \underline{\mathbf{M}} \underline{\phi}_{Ri} = \underline{\omega}_i \quad (5.82)$$

$$\underline{\phi}_{Li}^T \underline{\mathbf{D}} \underline{\phi}_{Ri} = \underline{\psi}_i \quad (5.83)$$

$$\underline{\phi}_{Li}^T \underline{\mathbf{K}} \underline{\phi}_{Ri} = \underline{\gamma}_i \quad (5.84)$$

5.7 Diagonalising structure-preserving equivalences (DSPEs)

Since the eigenvalues of system $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ are most usually calculated as the roots of

$$\det(\underline{\mathbf{D}} - \lambda \underline{\mathbf{K}}) = 0 \quad (5.85)$$

If these roots are all distinct, then it is always possible to find matrices $\{\underline{\Phi}_L, \underline{\Phi}_R\}$ such that $(\underline{\Phi}_L^T \underline{\mathbf{D}} \underline{\Phi}_R)$ and $(\underline{\Phi}_L^T \underline{\mathbf{K}} \underline{\Phi}_R)$ are diagonal. In all such cases, it is always possible to find some SPEs for the system such that $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ of equations (2.70), (2.71) and (2.72) are all diagonal. An algorithmic approach to determining diagonalising $\{\underline{\mathbf{T}}_L, \underline{\mathbf{T}}_R\}$ from $\{\underline{\Phi}_L, \underline{\Phi}_R\}$ is outlined in [44, 45]. In [46], a process is described by which the matrices $\{\underline{\mathbf{T}}_L, \underline{\mathbf{T}}_R\}$ are developed from a numerical solution of the differential equations outlined in [124]. A major motivation for this chapter is that even when it is not possible to find matrices $\{\underline{\Phi}_L, \underline{\Phi}_R\}$ such that $(\underline{\Phi}_L^T \underline{\mathbf{D}}_o \underline{\Phi}_R)$ and $(\underline{\Phi}_L^T \underline{\mathbf{K}}_o \underline{\Phi}_R)$ are diagonal, it may still be possible to determine diagonalising $\{\underline{\mathbf{T}}_L, \underline{\mathbf{T}}_R\}$. This happens particularly where there are pairs of repeated real roots.

The diagonalising SPEs are introduced in Chapter 2 section (2.6.1). The diagonalising transformations for the general damping system $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ are

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_D \\ \mathbf{K}_D & \mathbf{D}_D \end{bmatrix} \quad (5.86)$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{K}_D & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_D \end{bmatrix} \quad (5.87)$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} -\mathbf{D}_D & -\mathbf{M}_D \\ -\mathbf{M}_D & \mathbf{0} \end{bmatrix} \quad (5.88)$$

or equivalently

$$\underline{\mathbf{T}}_L^T \underline{\mathbf{M}} \underline{\mathbf{T}}_R = \underline{\mathbf{M}}_D \quad (5.89)$$

$$\underline{\mathbf{T}}_L^T \underline{\mathbf{D}} \underline{\mathbf{T}}_R = \underline{\mathbf{D}}_D \quad (5.90)$$

$$\underline{\mathbf{T}}_L^T \underline{\mathbf{K}} \underline{\mathbf{T}}_R = \underline{\mathbf{K}}_D \quad (5.91)$$

where $\{\underline{\mathbf{K}}_D, \underline{\mathbf{D}}_D, \underline{\mathbf{M}}_D\}$ are block diagonal matrices and $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$ represent the *LAMs* for the system whose coefficient matrices are the diagonal matrices $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$

$$\begin{aligned} \mathbf{K}_D &= \text{diag}(k_1, k_2, \dots, k_n) \\ \mathbf{D}_D &= \text{diag}(d_1, d_2, \dots, d_n) \\ \mathbf{M}_D &= \text{diag}(m_1, m_2, \dots, m_n) \end{aligned} \quad (5.92)$$

The following serve as a general definition for a pair of eigenvalues of a second-order system and the associated pairs of left and right eigenvectors.

$$\left[k_i \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} + d_i \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} + m_i \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \right] \begin{bmatrix} \mathbf{w}_{Ri} & \mathbf{x}_{Ri} \\ \mathbf{y}_{Ri} & \mathbf{z}_{Ri} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (5.93)$$

$$\begin{bmatrix} \mathbf{w}_{Li} & \mathbf{x}_{Li} \\ \mathbf{y}_{Li} & \mathbf{z}_{Li} \end{bmatrix}^T \left[k_i \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} + d_i \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} + m_i \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \right] = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (5.94)$$

more concisely

$$(k_i \underline{\mathbf{K}} + d_i \underline{\mathbf{D}} + m_i \underline{\mathbf{M}}) \underline{\mathbf{t}}_{Ri} = 0 \quad (5.95)$$

$$\underline{\mathbf{t}}_{Li}^T (k_i \underline{\mathbf{K}} + d_i \underline{\mathbf{D}} + m_i \underline{\mathbf{M}}) = 0 \quad (5.96)$$

Here, $\underline{\mathbf{t}}_{Ri}$ is a matrix of dimension $(2n \times 2)$ whereas \mathbf{w}_{Ri} is a column-vector of size n

and \underline{k}_i is a (2×2) matrix whilst k_i is a scalar. $\{\underline{\mathbf{t}}_{Li}, \underline{\mathbf{t}}_{Ri}\}$ will be partitioned as follows

$$\underline{\mathbf{t}}_{Li} = \begin{bmatrix} \mathbf{w}_{Li} & \mathbf{x}_{Li} \\ \mathbf{y}_{Li} & \mathbf{z}_{Li} \end{bmatrix} \quad (5.97)$$

$$\underline{\mathbf{t}}_{Ri} = \begin{bmatrix} \mathbf{w}_{Ri} & \mathbf{x}_{Ri} \\ \mathbf{y}_{Ri} & \mathbf{z}_{Ri} \end{bmatrix} \quad (5.98)$$

These partitions are related to the full matrices of the diagonalising transformation through

$$\mathbf{W}_R = [\mathbf{w}_{R1} \ \mathbf{w}_{R2} \ \dots \ \mathbf{w}_{RN}] \ , \ \mathbf{W}_L = [\mathbf{w}_{L1} \ \mathbf{w}_{L2} \ \dots \ \mathbf{w}_{LN}] \quad (5.99)$$

$$\mathbf{X}_R = [\mathbf{x}_{R1} \ \mathbf{x}_{R2} \ \dots \ \mathbf{x}_{RN}] \ , \ \mathbf{X}_L = [\mathbf{x}_{L1} \ \mathbf{x}_{L2} \ \dots \ \mathbf{x}_{LN}] \quad (5.100)$$

$$\mathbf{Y}_R = [\mathbf{y}_{R1} \ \mathbf{y}_{R2} \ \dots \ \mathbf{y}_{RN}] \ , \ \mathbf{Y}_L = [\mathbf{y}_{L1} \ \mathbf{y}_{L2} \ \dots \ \mathbf{y}_{LN}] \quad (5.101)$$

$$\mathbf{Z}_R = [\mathbf{z}_{R1} \ \mathbf{z}_{R2} \ \dots \ \mathbf{z}_{RN}] \ , \ \mathbf{Z}_L = [\mathbf{z}_{L1} \ \mathbf{z}_{L2} \ \dots \ \mathbf{z}_{LN}] \quad (5.102)$$

and the matrices $\{\mathbf{W}_L, \mathbf{X}_L, \mathbf{Y}_L, \mathbf{Z}_L, \mathbf{W}_R, \mathbf{X}_R, \mathbf{Y}_R, \mathbf{Z}_R\}$ collectively form $\{\underline{\mathbf{T}}_L, \underline{\mathbf{T}}_R\}$ as equations (5.86)-(5.88) indicate. It is evident that $\{\underline{\mathbf{t}}_{Li}, \underline{\mathbf{t}}_{Ri}\}$ hold the i^{th} and $(n+i)$ columns of the left and right diagonalising transformations $\{\underline{\mathbf{T}}_L, \underline{\mathbf{T}}_R\}$ respectively.

$$\underline{\mathbf{T}}_L \underline{\mathbf{e}}_i = \underline{\mathbf{t}}_{Li} \quad (5.103)$$

$$\underline{\mathbf{T}}_R \underline{\mathbf{e}}_i = \underline{\mathbf{t}}_{Ri} \quad (5.104)$$

where $\underline{\mathbf{e}}_i$ is a $(2n \times 2)$ matrix which is defined as

$$\underline{\mathbf{e}}_i = \begin{bmatrix} \mathbf{e}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{e}_i \end{bmatrix} \quad (5.105)$$

Here, the notation \mathbf{e}_i denotes the i^{th} column of the $(n \times n)$ identity matrix. Equations (5.86)-(5.88) can be written as

$$\begin{bmatrix} w_{Li} & x_{Li} \\ y_{Li} & z_{Li} \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} w_{Ri} & x_{Ri} \\ y_{Ri} & z_{Ri} \end{bmatrix} = \begin{bmatrix} 0 & k_i \\ k_i & d_i \end{bmatrix} \quad (5.106)$$

$$\begin{bmatrix} w_{Li} & x_{Li} \\ y_{Li} & z_{Li} \end{bmatrix}^T \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{bmatrix} w_{Ri} & x_{Ri} \\ y_{Ri} & z_{Ri} \end{bmatrix} = \begin{bmatrix} k_i & 0 \\ 0 & -m_i \end{bmatrix} \quad (5.107)$$

$$\begin{bmatrix} w_{Li} & x_{Li} \\ y_{Li} & z_{Li} \end{bmatrix}^T \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} w_{Ri} & x_{Ri} \\ y_{Ri} & z_{Ri} \end{bmatrix} = \begin{bmatrix} -d_i & -m_i \\ -m_i & 0 \end{bmatrix} \quad (5.108)$$

Simplifying equations (5.106)-(5.108) as

$$\underline{\mathbf{t}}_{Li}^T \underline{\mathbf{K}} \underline{\mathbf{t}}_{Ri} = k_i \quad (5.109)$$

$$\underline{\mathbf{t}}_{Li}^T \underline{\mathbf{D}} \underline{\mathbf{t}}_{Ri} = \underline{d}_i \quad (5.110)$$

$$\underline{\mathbf{t}}_{Li}^T \underline{\mathbf{M}} \underline{\mathbf{t}}_{Ri} = \underline{m}_i \quad (5.111)$$

These equations arise by post-multiplying each of equations (5.89)-(5.91) by $\underline{\mathbf{e}}_i$ and pre-multiplying each one by its transpose. The left and right diagonalising transformations $\{\underline{\mathbf{t}}_{Li}, \underline{\mathbf{t}}_{Ri}\}$ satisfy equations (5.95) and (5.96)

A new *homogeneous coordinates* definition for the eigenvalues and eigenvectors of a second-order system can be developed. Multiply both sides of equation (5.109) by k_i and multiply both sides of equation (5.110) by d_i and multiply both sides of equation (5.111) by m_i and then add the three equations. Observe that

$$\left(m_i \begin{bmatrix} 0 & k_i \\ k_i & d_i \end{bmatrix} + d_i \begin{bmatrix} k_i & 0 \\ 0 & -m_i \end{bmatrix} + k_i \begin{bmatrix} -d_i & -m_i \\ -m_i & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (5.112)$$

or equivalently

$$(m_i \underline{m}_i + d_i \underline{d}_i + k_i \underline{k}_i) = 0 \quad (5.113)$$

It follows immediately that if $\{k_i, d_i, m_i\}$ are the i^{th} diagonal entries of the diagonal matrices $\{\underline{\mathbf{K}}_D, \underline{\mathbf{D}}_D, \underline{\mathbf{M}}_D\}$ respectively, then the matrix $(m_i \underline{\mathbf{M}}_D + d_i \underline{\mathbf{D}}_D + k_i \underline{\mathbf{K}}_D)$ must have (at least) two zero singular values. It then follows that if equations (5.89)-(5.91) apply and if $\{\underline{\mathbf{T}}_L, \underline{\mathbf{T}}_R\}$ are both invertible, then $(m_i \underline{\mathbf{M}} + d_i \underline{\mathbf{D}} + k_i \underline{\mathbf{K}})$ must also have (at least) two zero singular values.

5.8 Derivatives of diagonalised damped system

The derivatives of the eigenvalues and eigenvectors of damped second-order systems provide additional challenges but there is obviously strong motivation for studying these. Cardani and Mantegazza [125] considered damping in the context of flutter problems and noted that the eigenvalues, eigenvectors and their derivatives become complex in general. Adhikari [126] derived exact expressions for the derivatives of complex eigenvalues and eigenvectors for systems having non-proportional viscous damping - avoiding the use of a state space representation of the equation of motion. Friswell and Adhikari [127] developed Nelson's method for symmetric non-proportionally damped systems with complex modes. Adhikari and Friswell [128] developed expressions for the first and second derivatives of complex eigensolutions of general asymmetric nonconservative systems.

In this section, we concentrate on the derivatives of three diagonal matrices, $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$. In effect, we are finding the derivatives of the eigenvalues of the general second-order system. However, this approach does not suffer from the eigenvalue derivatives becoming undefined in the presence of a single pair of identical real roots and it does not have any restriction to non-infinite eigenvalues.

Differentiate equations (5.89)-(5.91) to obtain:

$$\dot{\mathbf{T}}_L^T \mathbf{M} \mathbf{T}_R + \mathbf{T}_L^T \dot{\mathbf{M}} \mathbf{T}_R + \mathbf{T}_L^T \mathbf{M} \dot{\mathbf{T}}_R = \dot{\mathbf{M}}_D \quad (5.114)$$

$$\dot{\mathbf{T}}_L^T \mathbf{D} \mathbf{T}_R + \mathbf{T}_L^T \dot{\mathbf{D}} \mathbf{T}_R + \mathbf{T}_L^T \mathbf{D} \dot{\mathbf{T}}_R = \dot{\mathbf{D}}_D \quad (5.115)$$

$$\dot{\mathbf{T}}_L^T \mathbf{K} \mathbf{T}_R + \mathbf{T}_L^T \dot{\mathbf{K}} \mathbf{T}_R + \mathbf{T}_L^T \mathbf{K} \dot{\mathbf{T}}_R = \dot{\mathbf{K}}_D \quad (5.116)$$

Following a close parallel to the logic used in section 5.3, for some given i , multiply equation (5.114) by \underline{m}_i , multiply equation (5.115) by \underline{d}_i and multiply equation (5.116) by \underline{k}_i . Adding the three resulting equations and pre- and post-multiplying by $\underline{\mathbf{e}}_i^T$ and $\underline{\mathbf{e}}_i$, respectively, yields

$$\underline{\mathbf{t}}_{Li}^T (\underline{k}_i \underline{\dot{\mathbf{K}}} + \underline{d}_i \underline{\dot{\mathbf{D}}} + \underline{m}_i \underline{\dot{\mathbf{M}}}) \underline{\mathbf{t}}_{Ri} = (\underline{k}_i \underline{\dot{k}}_i + \underline{d}_i \underline{\dot{d}}_i + \underline{m}_i \underline{\dot{m}}_i) \quad (5.117)$$

where the definitions of $\{\underline{k}_i, \underline{d}_i, \underline{m}_i\}$ are

$$\underline{k}_i = \begin{bmatrix} -\underline{d}_i & -\underline{m}_i \\ -\underline{m}_i & 0 \end{bmatrix}, \quad \underline{d}_i = \begin{bmatrix} \underline{k}_i & 0 \\ 0 & -\underline{m}_i \end{bmatrix}, \quad \underline{m}_i = \begin{bmatrix} 0 & \underline{k}_i \\ \underline{k}_i & \underline{d}_i \end{bmatrix} \quad (5.118)$$

In determining equation (5.117), equations (5.95), (5.96), (5.103) and (5.104) were applied. Now, since all quantities on the LHS of equation (5.117) are known along with $\{\underline{k}_i, \underline{d}_i, \underline{m}_i\}$, it is clear that there are four scalar equations in equation (5.117) involving the three unknowns $\{\underline{\dot{k}}_i, \underline{\dot{d}}_i, \underline{\dot{m}}_i\}$. It is immediately obvious, by symmetry that, at most, only three scalar equations are independent. These three resulting equations can be rearranged to have the form:

$$\begin{bmatrix} \underline{d}_i & -\underline{k}_i & 0 \\ \underline{m}_i & 0 & -\underline{k}_i \\ 0 & \underline{m}_i & -\underline{d}_i \end{bmatrix} \begin{bmatrix} \underline{\dot{k}}_i \\ \underline{\dot{d}}_i \\ \underline{\dot{m}}_i \end{bmatrix} = \begin{bmatrix} \diamond \\ \diamond \\ \diamond \end{bmatrix} \quad (5.119)$$

where, once again, the diamond symbol, \diamond , is being used to indicate a known scalar quantity. The (3×3) matrix on the left hand side of equation (5.119) always has one zero singular value. The fact that $\{\underline{\dot{k}}_i, \underline{\dot{d}}_i, \underline{\dot{m}}_i\}$ is not uniquely defined is consistent with the fact that the scaling of the diagonal entries themselves is not unique. We found something similar for the case of an undamped system. By choosing to set the scaling

$$\underline{k}_i^2 + \underline{d}_i^2 + \underline{m}_i^2 = 1 \quad \text{for all } i \quad (5.120)$$

then equation (5.119) becomes

$$\begin{bmatrix} d_i & -k_i & 0 \\ m_i & 0 & -k_i \\ 0 & m_i & -d_i \\ k_i & d_i & m_i \end{bmatrix} \begin{bmatrix} \dot{k}_i \\ \dot{d}_i \\ \dot{m}_i \end{bmatrix} = \begin{bmatrix} \diamond \\ \diamond \\ \diamond \\ 0 \end{bmatrix} \quad (5.121)$$

Rather than eliminate any one row from equation (5.119) when forming equation (5.121), it is recommended to leave four scalar equations in equation (5.121) and to solve this using the pseudo-inverse. The columns of the (4×3) matrix in equation (5.121) are mutually orthogonal and thus very stable solutions can be found for the three rates of change.

5.8.1 *Eigenvalues and their derivatives from diagonalised systems*

Eigenvalues and their derivatives can be calculated from $\{k_i, d_i, m_i\}$ and their derivatives $\{\dot{k}_i, \dot{d}_i, \dot{m}_i\}$ using the following equation

$$k_i + d_i \lambda + m_i \lambda^2 = 0 \quad (5.122)$$

Then, a pair of eigenvalues is represented by

$$\lambda = \frac{-d_i \pm \sqrt{d_i^2 - 4m_i k_i}}{2m_i} \quad (5.123)$$

Differentiating equation (5.122) with respect to θ gives an expression for the eigenvalue derivatives (see Appendix D).

$$\dot{\lambda} = \frac{-(\dot{k}_i + \dot{d}_i \lambda + \dot{m}_i \lambda^2)}{(d_i + 2m_i \lambda)} \quad (5.124)$$

For any three scalars $\{k_i, d_i, m_i\}$ equations (5.71) and (5.93) are two equivalent statements (further details are in Appendix C).

5.9 Derivatives of the diagonalising transformations

In this section, we follow a close parallel to the development of section 5.5. We begin with equations (5.95) and (5.96) as definitions of the left and right “eigenvectors” in the sense that they contain the eigenvector information associated with the pair of eigenvalues which are roots of equation (5.122). As was the case with Section 5.5, we shall see that no other eigenvector information is required to find the required derivatives of $\{\underline{\dot{\mathbf{t}}}_{Li}, \underline{\dot{\mathbf{t}}}_{Ri}\}$.

Differentiating equations (5.95) and (5.96) with respect to scalar parameter σ produces

$$(k_i \underline{\mathbf{K}} + d_i \underline{\mathbf{D}} + m_i \underline{\mathbf{M}}) \underline{\dot{\mathbf{t}}}_{Ri} = -\frac{d}{d\sigma} (k_i \underline{\mathbf{K}} + d_i \underline{\mathbf{D}} + m_i \underline{\mathbf{M}}) \underline{\mathbf{t}}_{Ri} \quad (5.125)$$

$$\underline{\dot{\mathbf{t}}}_{Li}^T (k_i \underline{\mathbf{K}} + d_i \underline{\mathbf{D}} + m_i \underline{\mathbf{M}}) = -\underline{\mathbf{t}}_{Li}^T \frac{d}{d\sigma} (k_i \underline{\mathbf{K}} + d_i \underline{\mathbf{D}} + m_i \underline{\mathbf{M}}) \quad (5.126)$$

The right hand sides of the above equations are known. We now follow a procedure almost identical to that of section 5.5 where the derivatives of the eigenvectors of an undamped system were derived. First, recall the definition of $\underline{\mathbf{e}}_i$ from equation (5.105). In the same way that $\{\underline{\mathbf{e}}_i, \underline{\mathbf{S}}_i\}$ together span n -space, we define a new $(2n \times (2n-2))$ matrix, $\underline{\mathbf{S}}_i$, such that $\{\underline{\mathbf{e}}_i, \underline{\mathbf{S}}_i\}$ together span $2n$ -space. A logical format for $\underline{\mathbf{S}}_i$ is this

$$\underline{\mathbf{S}}_i = \begin{bmatrix} \underline{\mathbf{S}}_i & \mathbf{0} \\ \mathbf{0} & \underline{\mathbf{S}}_i \end{bmatrix} \quad (5.127)$$

The developed method is to generate two orthogonal matrices $\{\underline{\mathbf{Q}}_{Li}, \underline{\mathbf{Q}}_{Ri}\}$ which are orthogonal to the $\{\underline{\mathbf{t}}_{Li}, \underline{\mathbf{t}}_{Ri}\}$. The $\{\underline{\mathbf{Q}}_{Li}, \underline{\mathbf{Q}}_{Ri}\}$ matrices can represent left and right Householder Reflections and they satisfy the following equations

$$\underline{\mathbf{Q}}_{Li} \underline{\mathbf{t}}_{Li} = \underline{\mathbf{e}}_i \alpha_{Li} \quad (5.128)$$

$$\underline{\mathbf{Q}}_{Ri} \underline{\mathbf{t}}_{Ri} = \underline{\mathbf{e}}_i \alpha_{Ri} \quad (5.129)$$

Then, $\{\underline{\mathbf{Q}}_{Li}, \underline{\mathbf{Q}}_{Ri}\}$ satisfy the following equation

$$(\underline{\mathbf{S}}_i^T \underline{\mathbf{Q}}_{Li} \underline{\mathbf{t}}_{Li}) = \mathbf{0} = (\underline{\mathbf{S}}_i^T \underline{\mathbf{Q}}_{Ri} \underline{\mathbf{t}}_{Ri}) \quad (5.130)$$

where $\{\underline{\mathbf{a}}_{Li}, \underline{\mathbf{a}}_{Ri}\}$ are any two arbitrary (2×2) matrices which can be defined as

$$\underline{\mathbf{a}}_{Li} = \begin{bmatrix} \mathbf{a}_{L1} & \mathbf{a}_{L3} \\ \mathbf{a}_{L4} & \mathbf{a}_{L2} \end{bmatrix}, \quad \underline{\mathbf{a}}_{Ri} = \begin{bmatrix} \mathbf{a}_{R1} & \mathbf{a}_{R3} \\ \mathbf{a}_{R4} & \mathbf{a}_{R2} \end{bmatrix} \quad (5.131)$$

The left and right diagonalising transformations $\{\underline{\mathbf{t}}_{Li}, \underline{\mathbf{t}}_{Ri}\}$ comprise two columns each $\underline{\mathbf{t}}_{Li} = [\underline{\mathbf{t}}_{Li1} \quad \underline{\mathbf{t}}_{Li2}]$ and $\underline{\mathbf{t}}_{Ri} = [\underline{\mathbf{t}}_{Ri1} \quad \underline{\mathbf{t}}_{Ri2}]$ respectively.

The two orthogonal matrices $\{\underline{\mathbf{Q}}_{Li}, \underline{\mathbf{Q}}_{Ri}\}$ can be calculated by regarding each column in the left and right diagonalising transformation $\{\underline{\mathbf{t}}_{Li}, \underline{\mathbf{t}}_{Ri}\}$ using the following equations

$$\begin{aligned} \underline{\mathbf{Q}}_{Li} &= \underline{\mathbf{Q}}_{Li2} \underline{\mathbf{Q}}_{Li1} = (\underline{\mathbf{I}} - 2\underline{\mathbf{u}}_{L2} \underline{\mathbf{u}}_{L2}^T) (\underline{\mathbf{I}} - 2\underline{\mathbf{u}}_{L1} \underline{\mathbf{u}}_{L1}^T) \\ \underline{\mathbf{Q}}_{Ri} &= \underline{\mathbf{Q}}_{Ri2} \underline{\mathbf{Q}}_{Ri1} = (\underline{\mathbf{I}} - 2\underline{\mathbf{u}}_{R2} \underline{\mathbf{u}}_{R2}^T) (\underline{\mathbf{I}} - 2\underline{\mathbf{u}}_{R1} \underline{\mathbf{u}}_{R1}^T) \end{aligned} \quad (5.132)$$

$\{\underline{\mathbf{Q}}_{Li1}, \underline{\mathbf{Q}}_{Li2}\}$, $\{\underline{\mathbf{Q}}_{Ri1}, \underline{\mathbf{Q}}_{Ri2}\}$ are symmetric orthogonal matrices. Vectors $\{\underline{\mathbf{u}}_{L1}, \underline{\mathbf{u}}_{R1}\}$ and $\{\underline{\mathbf{u}}_{L2}, \underline{\mathbf{u}}_{R2}\}$ are unit vectors. This calculation can be divided into two steps: the first

step is to calculate the $\{\underline{\mathbf{Q}}_{Li1}, \underline{\mathbf{Q}}_{Ri1}\}$ based on the first column in $\{\underline{\mathbf{t}}_{Li}, \underline{\mathbf{t}}_{Ri}\}$ and the second step is to calculate the $\{\underline{\mathbf{Q}}_{Li2}, \underline{\mathbf{Q}}_{Ri2}\}$ based on the second column in $\{\underline{\mathbf{t}}_{Li}, \underline{\mathbf{t}}_{Ri}\}$.

To find the orthogonal matrices $\{\underline{\mathbf{Q}}_{Li1}, \underline{\mathbf{Q}}_{Ri1}\}$ in equation (5.132) based on the first column in $\{\underline{\mathbf{t}}_{Li}, \underline{\mathbf{t}}_{Ri}\}$, the vectors $\{\underline{\mathbf{u}}_{L1}, \underline{\mathbf{u}}_{R1}\}$ can be calculated from the columns $\{\underline{\mathbf{t}}_{Li1}, \underline{\mathbf{t}}_{Ri1}\}$ as

$$\begin{aligned}\underline{\mathbf{u}}_{L1} &= \underline{\mathbf{t}}_{Li1} - \underline{\mathbf{e}}_{i1} \alpha_{L1} \quad , \quad \underline{\mathbf{u}}_{L1} = \frac{\underline{\mathbf{u}}_{L1}}{\|\underline{\mathbf{u}}_{L1}\|} \\ \underline{\mathbf{u}}_{R1} &= \underline{\mathbf{t}}_{Ri1} - \underline{\mathbf{e}}_{i1} \alpha_{R1} \quad , \quad \underline{\mathbf{u}}_{R1} = \frac{\underline{\mathbf{u}}_{R1}}{\|\underline{\mathbf{u}}_{R1}\|}\end{aligned}\tag{5.133}$$

where $\underline{\mathbf{e}}_{i1}$ represents the first column of the matrix $\underline{\mathbf{e}}_i$ and the scalar $\{\alpha_{L1}, \alpha_{R1}\}$ can be calculated from the following expression

$$\alpha_{L1} = \frac{\|\underline{\mathbf{t}}_{Li1}\|}{\|\underline{\mathbf{e}}_{i1}\|} \quad , \quad \alpha_{R1} = \frac{\|\underline{\mathbf{t}}_{Ri1}\|}{\|\underline{\mathbf{e}}_{i1}\|}\tag{5.134}$$

Then, $\{\underline{\mathbf{Q}}_{Li1}, \underline{\mathbf{Q}}_{Ri1}\}$ can be calculated using the following equations

$$\begin{aligned}\underline{\mathbf{Q}}_{Li1} &= (\underline{\mathbf{I}} - 2\underline{\mathbf{u}}_{L1}\underline{\mathbf{u}}_{L1}^T) \\ \underline{\mathbf{Q}}_{Ri1} &= (\underline{\mathbf{I}} - 2\underline{\mathbf{u}}_{R1}\underline{\mathbf{u}}_{R1}^T)\end{aligned}\tag{5.135}$$

To find orthogonal matrices $\{\underline{\mathbf{Q}}_{Li2}, \underline{\mathbf{Q}}_{Ri2}\}$ based on the second column in $\{\underline{\mathbf{t}}_{Li}, \underline{\mathbf{t}}_{Ri}\}$, the vectors $\{\underline{\mathbf{u}}_{L2}, \underline{\mathbf{u}}_{R2}\}$ can be calculated using the following expressions

$$\underline{\mathbf{t}}_{Lni} = (\underline{\mathbf{I}} - 2\underline{\mathbf{u}}_{L1}\underline{\mathbf{u}}_{L1}^T)\underline{\mathbf{t}}_{Li} \quad , \quad \underline{\mathbf{t}}_{Rni} = (\underline{\mathbf{I}} - 2\underline{\mathbf{u}}_{R1}\underline{\mathbf{u}}_{R1}^T)\underline{\mathbf{t}}_{Ri}\tag{5.136}$$

where $\{\underline{\mathbf{t}}_{Lni}, \underline{\mathbf{t}}_{Rni}\}$ comprise two columns each $\underline{\mathbf{t}}_{Lni} = \begin{bmatrix} \underline{\mathbf{t}}_{Lni1} & \underline{\mathbf{t}}_{Lni2} \end{bmatrix}$, $\underline{\mathbf{t}}_{Rni} = \begin{bmatrix} \underline{\mathbf{t}}_{Rni1} & \underline{\mathbf{t}}_{Rni2} \end{bmatrix}$.

The scalars $\{\alpha_{L3}, \alpha_{R3}\}$ are selected to be the i^{th} entry in the columns $\{\underline{\mathbf{t}}_{Lni1}, \underline{\mathbf{t}}_{Rni1}\}$ and

the scalars $\{\alpha_{L4}, \alpha_{R4}\}$ are selected to be the i^{th} entry in the columns $\{\underline{\mathbf{t}}_{Lni2}, \underline{\mathbf{t}}_{Rni2}\}$

respectively. Then, zero the i^{th} entry of columns $\{\underline{\mathbf{t}}_{Lni2}, \underline{\mathbf{t}}_{Rni2}\}$. Using the following

expression to calculate α_{L2}

$$\alpha_{L2} = \frac{\|\underline{\mathbf{T}}_{Lni2}\|}{\|\underline{\mathbf{e}}_{i2}\|}, \quad \alpha_{R2} = \frac{\|\underline{\mathbf{T}}_{Rni2}\|}{\|\underline{\mathbf{e}}_{i2}\|} \quad (5.137)$$

The vectors $\{\underline{\mathbf{u}}_{L2}, \underline{\mathbf{u}}_{R2}\}$ are obtained

$$\begin{aligned} \underline{\mathbf{u}}_{L2} &= \underline{\mathbf{t}}_{Lni2} - \underline{\mathbf{e}}_{i2} \alpha_{L2} \quad , \quad \underline{\mathbf{u}}_{L2} = \frac{\underline{\mathbf{u}}_{L2}}{\|\underline{\mathbf{u}}_{L2}\|} \\ \underline{\mathbf{u}}_{R2} &= \underline{\mathbf{t}}_{Rni2} - \underline{\mathbf{e}}_{i2} \alpha_{R2} \quad , \quad \underline{\mathbf{u}}_{R2} = \frac{\underline{\mathbf{u}}_{R2}}{\|\underline{\mathbf{u}}_{R2}\|} \end{aligned} \quad (5.138)$$

Then, $\{\underline{\mathbf{Q}}_{Li2}, \underline{\mathbf{Q}}_{Ri2}\}$ can be calculated using the following equations

$$\begin{aligned} \underline{\mathbf{Q}}_{Li2} &= (\underline{\mathbf{I}} - 2\underline{\mathbf{u}}_{L2}\underline{\mathbf{u}}_{L2}^T) \\ \underline{\mathbf{Q}}_{Ri2} &= (\underline{\mathbf{I}} - 2\underline{\mathbf{u}}_{R2}\underline{\mathbf{u}}_{R2}^T) \end{aligned} \quad (5.139)$$

The left and right diagonalising transformation derivatives $\{\underline{\mathbf{i}}_{Li}, \underline{\mathbf{i}}_{Ri}\}$ can be written as

$$\underline{\mathbf{i}}_{Li} = (\underline{\mathbf{Q}}_{Li}^T \underline{\mathbf{S}}_i) \underline{\mathbf{g}}_{Li} + \underline{\mathbf{t}}_{Li} \beta_{Li} \quad (5.140)$$

$$\underline{\mathbf{i}}_{Ri} = (\underline{\mathbf{Q}}_{Ri}^T \underline{\mathbf{S}}_i) \underline{\mathbf{g}}_{Ri} + \underline{\mathbf{t}}_{Ri} \beta_{Ri} \quad (5.141)$$

where $\{\beta_{Li}, \beta_{Ri}\}$ are (2×2) matrices and where $\{\underline{\mathbf{g}}_{Li}, \underline{\mathbf{g}}_{Ri}\}$ are matrices of dimension

$(2(n-1) \times 2)$ which can be determined uniquely by substituting for $\{\underline{\mathbf{i}}_{Li}, \underline{\mathbf{i}}_{Ri}\}$ in

equation (5.125), (5.126), respectively, using equations (5.140) and (5.141). Terms

involving the unknown (2×2) matrix quantities, $\{\beta_{Li}, \beta_{Ri}\}$, vanish naturally and the

result in each case is an overdetermined but consistent set of equations which can be solved directly and stably using a pseudo-inverse.

It remains only to state how the two (2×2) matrices $\{\underline{\beta}_{Li}, \underline{\beta}_{Ri}\}$ should be found.

$$\underline{\beta}_{Li} = \begin{bmatrix} \beta_{L11} & \beta_{L12} \\ \beta_{L21} & \beta_{L22} \end{bmatrix}, \quad \underline{\beta}_{Ri} = \begin{bmatrix} \beta_{R11} & \beta_{R12} \\ \beta_{R21} & \beta_{R22} \end{bmatrix} \quad (5.142)$$

Much of the requisite information is present through differentiating equations (5.109)-(5.111).

$$\underline{\dot{\mathbf{t}}}_{Li}^T \underline{\mathbf{K}} \underline{\mathbf{t}}_{Ri} + \underline{\mathbf{t}}_{Li}^T \underline{\mathbf{K}} \underline{\dot{\mathbf{t}}}_{Ri} = \underline{\dot{k}}_i - \underline{\mathbf{t}}_{Li}^T \underline{\dot{\mathbf{K}}} \underline{\mathbf{t}}_{Ri} = \underline{\mathbf{h}}_{ki} \quad (5.143)$$

$$\underline{\mathbf{t}}_{Li}^T \underline{\mathbf{D}} \underline{\mathbf{t}}_{Ri} + \underline{\mathbf{t}}_{Li}^T \underline{\mathbf{D}} \underline{\dot{\mathbf{t}}}_{Ri} = \underline{\dot{d}}_i - \underline{\mathbf{t}}_{Li}^T \underline{\dot{\mathbf{D}}} \underline{\mathbf{t}}_{Ri} = \underline{\mathbf{h}}_{di} \quad (5.144)$$

$$\underline{\mathbf{t}}_{Li}^T \underline{\mathbf{M}} \underline{\mathbf{t}}_{Ri} + \underline{\mathbf{t}}_{Li}^T \underline{\mathbf{M}} \underline{\dot{\mathbf{t}}}_{Ri} = \underline{\dot{m}}_i - \underline{\mathbf{t}}_{Li}^T \underline{\dot{\mathbf{M}}} \underline{\mathbf{t}}_{Ri} = \underline{\mathbf{h}}_{mi} \quad (5.145)$$

The derivatives in the right hand sides of equations (5.143)-(5.145) $\{\underline{\mathbf{h}}_{ki}, \underline{\mathbf{h}}_{di}, \underline{\mathbf{h}}_{mi}\}$ are (2×2) matrices that are known (see section 5.8) as are the derivatives of the LAMs $\{\underline{\dot{\mathbf{K}}}, \underline{\dot{\mathbf{D}}}, \underline{\dot{\mathbf{M}}}\}$. When equations (5.140) and (5.141) are used to substitute for $\{\underline{\mathbf{t}}_{Li}, \underline{\mathbf{t}}_{Ri}\}$, equations (5.143)-(5.145) yield 12 equations with only 8 unknowns (the 8 scalar entries of $\{\underline{\beta}_{Li}, \underline{\beta}_{Ri}\}$).

$$\begin{bmatrix} 0 & 0 & k_i & 0 & 0 & 0 & k_i & 0 \\ k_i & 0 & d_i & 0 & 0 & 0 & 0 & k_i \\ 0 & 0 & 0 & k_i & k_i & 0 & d_i & 0 \\ 0 & k_i & 0 & d_i & 0 & k_i & 0 & d_i \\ k_i & 0 & 0 & 0 & k_i & 0 & 0 & 0 \\ 0 & 0 & -m_i & 0 & 0 & k_i & 0 & 0 \\ 0 & k_i & 0 & 0 & 0 & 0 & -m_i & 0 \\ 0 & 0 & 0 & -m_i & 0 & 0 & 0 & -m_i \\ -d_i & 0 & -m_i & 0 & -d_i & 0 & -m_i & 0 \\ -m_i & 0 & 0 & 0 & 0 & -d_i & 0 & -m_i \\ 0 & -d_i & 0 & -m_i & -m_i & 0 & 0 & 0 \\ 0 & -m_i & 0 & 0 & 0 & -m_i & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta_{R11} \\ \beta_{R12} \\ \beta_{R21} \\ \beta_{R22} \\ \beta_{L11} \\ \beta_{L12} \\ \beta_{L21} \\ \beta_{L22} \end{bmatrix} = \begin{bmatrix} h_{m11} \\ h_{m12} \\ h_{m21} \\ h_{m22} \\ h_{d11} \\ h_{d12} \\ h_{d21} \\ h_{d22} \\ h_{k11} \\ h_{k12} \\ h_{k21} \\ h_{k22} \end{bmatrix} \quad (5.146)$$

or equivalently

$$\mathbf{W}\beta_i = \mathbf{h}_i \quad (5.147)$$

where \mathbf{W} is a (12×8) matrix with rank 6, β_i is a vector of 8 unknowns and \mathbf{h}_i is a vector of 12 known values. In fact, only 6 of these equations are independent. When the system matrices are symmetric, forcing $\underline{\beta}_{Li} = \underline{\beta}_{Ri}$ is always possible and this is sufficient to determine $\{\underline{\beta}_{Li}, \underline{\beta}_{Ri}\}$ uniquely. When the system matrices are not symmetric, some arbitrary choice must be made in the determination of these quantities. A reasonably general approach (which is consistent with the symmetric case) is to impose the constraint:

$$\text{diag}(\underline{\beta}_{Li}^T \underline{\beta}_{Li}) = \text{diag}(\underline{\beta}_{Ri}^T \underline{\beta}_{Ri}) \quad (5.148)$$

The solution of the vector β_i in equation (5.147) can be achieved from singular value decomposition of the matrix \mathbf{W}

$$[\mathbf{u} \Sigma \mathbf{v}] = \text{svd}(\mathbf{W}) \quad (5.149)$$

where $\{\mathbf{u}, \mathbf{v}\}$ have dimensions (12×12) , (8×8) , respectively, are two orthogonal matrices $(\mathbf{u}^T \mathbf{u} = \mathbf{I})$, $(\mathbf{v}^T \mathbf{v} = \mathbf{I})$ and Σ is a diagonal matrix with two zero singular values. The matrix \mathbf{W} can be written as

$$\mathbf{W} = \mathbf{u} \Sigma \mathbf{v}^T = [\mathbf{u}_1 \quad \mathbf{u}_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} [\mathbf{v}_1 \quad \mathbf{v}_2]^T \quad (5.150)$$

where the number of columns of $\{\mathbf{u}_1, \mathbf{v}_1\}$ equals rank of the matrix \mathbf{W} , $\mathbf{u}_1^T \mathbf{u}_1 = \mathbf{I}$ and $(\mathbf{v}_1^T \mathbf{v}_1 = \mathbf{I})$. The vector β_i can be expressed in the following form

$$\beta_i = (\mathbf{w} + \mathbf{z}) \quad (5.151)$$

where \mathbf{w} can be calculated using this equation

$$\mathbf{w} = \mathbf{v}_1 \mathbf{r} = \mathbf{v}_1 \Sigma_1^{-1} \mathbf{x} = \mathbf{v}_1 \Sigma_1^{-1} \mathbf{u}_1^T \mathbf{h}_i \quad (5.152)$$

and the vector \mathbf{z} can be calculated using the following expression

$$\mathbf{z} = \mathbf{v}_2 \mathbf{b} \quad (5.153)$$

in which \mathbf{z} satisfies $\mathbf{W}\mathbf{z} = \mathbf{0}$. The vector \mathbf{b} is a random vector, has a dimension equal to the number of zero singular values.

5.9.1 Eigenvectors and their derivatives from diagonalising transformations

Comparing equations (5.79)-(5.81) with equations (5.106)-(5.108) yields

$$\begin{bmatrix} \lambda_i \psi_i & \mathbf{0} \\ \mathbf{0} & \lambda_i^* \psi_i^* \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & k_i \\ k_i & d_i \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad (5.154)$$

$$\begin{bmatrix} \psi_i & \mathbf{0} \\ \mathbf{0} & \psi_i^* \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^T \begin{bmatrix} k_i & 0 \\ 0 & -m_i \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad (5.155)$$

$$\begin{bmatrix} \psi_i / \lambda_i & \mathbf{0} \\ \mathbf{0} & \psi_i^* / \lambda_i^* \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^T \begin{bmatrix} -d_i & -m_i \\ -m_i & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad (5.156)$$

$$\underline{\omega}_i = \underline{\mathbf{a}}_i^T \underline{m}_i \underline{\mathbf{a}}_i \quad (5.157)$$

$$\underline{\psi}_i = \underline{\mathbf{a}}_i^T \underline{d}_i \underline{\mathbf{a}}_i \quad (5.158)$$

$$\underline{\gamma}_i = \underline{\mathbf{a}}_i^T \underline{k}_i \underline{\mathbf{a}}_i \quad (5.159)$$

Equations (5.157)-(5.159) show that the (2×2) matrix $\underline{\mathbf{a}}_i$ is the eigenvector of the system $\{k_i, d_i, m_i\}$. Scaling the vectors in matrix $\underline{\mathbf{a}}_i$ yields

$$\underline{\mathbf{a}}_i = \underline{\mathbf{a}}_i / \text{diag}([\mathbf{a}_{11} \quad \mathbf{a}_{12}]) \quad (5.160)$$

The matrix $\underline{\mathbf{a}}_i$ can be written as

$$\underline{\mathbf{a}}_i = \begin{bmatrix} 1 & 1 \\ \lambda_i & \lambda_i^* \end{bmatrix} \quad (5.161)$$

The left and right columns i and $(n+i)$ of the eigenvector matrix of the original system can be calculated from the left and right columns i and $(n+i)$ of the diagonalising transformations as

$$\begin{bmatrix} \mathbf{u}_R & \mathbf{u}_R^* \\ \lambda_i \mathbf{u}_R & \lambda_i^* \mathbf{u}_R^* \end{bmatrix} = \begin{bmatrix} w_{Ri} & x_{Ri} \\ y_{Ri} & z_{Ri} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \lambda_i & \lambda_i^* \end{bmatrix} \quad (5.162)$$

$$\begin{bmatrix} \mathbf{u}_{Li} & \mathbf{u}_{Li}^* \\ \lambda_i \mathbf{u}_{Li} & \lambda_i^* \mathbf{u}_{Li}^* \end{bmatrix} = \begin{bmatrix} w_{Li} & x_{Li} \\ y_{Li} & z_{Li} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \lambda_i & \lambda_i^* \end{bmatrix} \quad (5.163)$$

Equivalently, we can write

$$\underline{\phi}_{Ri} = \underline{\mathbf{t}}_{Ri} \underline{\mathbf{a}}_i \quad (5.164)$$

$$\underline{\phi}_{Li} = \underline{\mathbf{t}}_{Li} \underline{\mathbf{a}}_i \quad (5.165)$$

Differentiating equations (5.164) and (5.165) with respect to σ delivers a new expression for the eigenvector derivatives (see Appendix D) as

$$\frac{d\underline{\phi}_{Ri}}{d\sigma} = \underline{\dot{\phi}}_{Ri} = \underline{\dot{\mathbf{t}}}_{Ri} \underline{\mathbf{a}}_i + \underline{\mathbf{t}}_{Ri} \underline{\dot{\mathbf{a}}}_i \quad (5.166)$$

$$\frac{d\underline{\phi}_{Li}}{d\sigma} = \underline{\dot{\phi}}_{Li} = \underline{\dot{\mathbf{t}}}_{Li} \underline{\mathbf{a}}_i + \underline{\mathbf{t}}_{Li} \underline{\dot{\mathbf{a}}}_i \quad (5.167)$$

where the derivative of $\underline{\mathbf{a}}_i$ can be written as

$$\underline{\dot{\mathbf{a}}}_i = \begin{bmatrix} 0 & 0 \\ \dot{\lambda}_i & \dot{\lambda}_i^* \end{bmatrix} \quad (5.168)$$

5.10 Examples

Four examples are presented here. For three of the cases, the systems are symmetric and have dimension (3×3) . In the other case, the system is undamped and non-symmetric with dimension (2×2) . All systems are described in terms of the scalar

parameter, σ , through equations (5.1)-(5.3) where $\{\mathbf{M}_o, \mathbf{D}_o, \mathbf{K}_o\}$ and $\{\mathbf{M}_1, \mathbf{D}_1, \mathbf{K}_1\}$ are given explicitly.

5.10.1 Example 1: A pair of identical real roots

In this case, there is a pair of identical real roots. Here, the point $\sigma=0$ coincides exactly with a point on the root-locus plot where two repeated real roots are just about to turn into two complex conjugate roots or vice-versa. Established methods indicate (correctly) that eigenvalue derivatives are undefined in such cases. However, the coefficients of the corresponding quadratic polynomial vary smoothly and are well-behaved. The *eigenvector scaling* is selected such that $(m_i^2 + d_i^2 + k_i^2) \equiv 1$ for all i .

$$\mathbf{M}_o = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 5 & 1 \\ 0 & 1 & 9 \end{bmatrix}, \quad \mathbf{K}_o = \begin{bmatrix} 800 & -300 & 0 \\ -300 & 900 & 50 \\ 0 & 50 & 1200 \end{bmatrix}, \quad \mathbf{M}_1 = \mathbf{K}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{D}_o = \begin{bmatrix} 26 & 10 & 24 \\ 10 & 18 & 15 \\ 24 & 15 & 40 \end{bmatrix} + \theta_{ref} \mathbf{D}_1, \quad \mathbf{D}_1 = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}, \quad \theta_{ref} = 13.4120370573992091$$

Table 5-1 summarises the values $\{m_i, d_i, k_i\}$ for each of the three pairs of modes as well as $\{\dot{m}_i, \dot{d}_i, \dot{k}_i\}$. Pairs of eigenvalues can be calculated from $\{m_i, d_i, k_i\}$ using $k_i + d_i\lambda + m_i\lambda^2 = 0$. The following eigenvalues are found:

$$\lambda = \begin{bmatrix} -1.5862 \pm 14.1363i \\ -1.4654 \pm 12.2227i \\ -12.1341 \pm 0i \end{bmatrix}$$

Figure 5-1 shows the root locus for the pair of repeated real roots for positive and negative values of σ close to $\sigma=0$.

	Mode Pair 1	Mode Pair 2	Mode Pair 3
m_i	4.9413e-3	6.5974e-3	6.7012e-3
d_i	1.5678e-2	1.9336e-2	1.6263e-1
k_i	9.9986e-1	9.9979e-1	9.8666e-1
\dot{m}_i	6.278e-6	-6.254e-6	-9.165e-6
\dot{d}_i	0.3514e-3	0.6544e-3	0.6149e-2
\dot{k}_i	-5.54e-6	-1.261e-5	-0.1013e-2

Table 5-1 : The diagonalised system and its rate of change

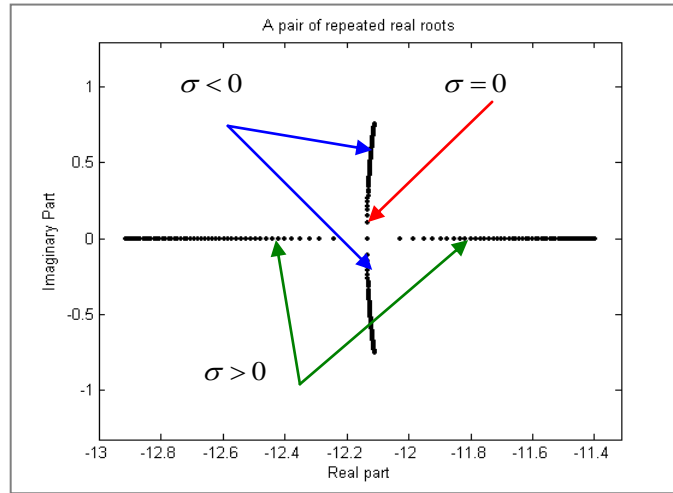


Figure 5-1 : Root locus for one pair of roots

5.10.2 Example 2: Singular Mass matrix

In this case, the mass matrix, M_o , is singular. This is deliberately selected to be a problematic case for the conventional eigenvalue solutions because one eigenvalue is infinity.

$$\mathbf{M}_o = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 2 \end{bmatrix}, \quad \mathbf{M}_1 = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\mathbf{K}_o = \begin{bmatrix} 800 & -300 & 0 \\ -300 & 900 & 50 \\ 0 & 50 & 1200 \end{bmatrix}, \quad \mathbf{K}_1 = \begin{bmatrix} 29 & 26 & -14 \\ 26 & 29 & -24 \\ -14 & -24 & 36 \end{bmatrix}$$

$$\mathbf{D}_o = \begin{bmatrix} 26 & 10 & 24 \\ 10 & 18 & 15 \\ 24 & 15 & 40 \end{bmatrix}, \quad \mathbf{D}_1 = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

The eigenvalues for the system are shown below

$$\lambda = \begin{bmatrix} -1.6401 \pm 11.1827i \\ -18.5730 \\ -5.4110 \pm 22.0597i \\ \text{Inf} \end{bmatrix}$$

Since the mass matrix is singular, equations (5.86)-(5.88) cannot be used easily to determine the diagonalised system. We take two linear combinations of the *LAMs*.

$$\underline{\mathbf{F}} = a\underline{\mathbf{M}}_o + b\underline{\mathbf{D}}_o + c\underline{\mathbf{K}}_o \quad (5.169)$$

$$\underline{\mathbf{G}} = d\underline{\mathbf{M}}_o + e\underline{\mathbf{D}}_o + f\underline{\mathbf{K}}_o \quad (5.170)$$

where $\{a, b, c, d, e, f\}$ are selected scalars and $\underline{\mathbf{G}}$ is an invertible matrix. The matrix of eigenvectors, \mathbf{X} , diagonalises all three LAMs in the sense that $\mathbf{X}^T \underline{\mathbf{M}}_o \mathbf{X}$, $\mathbf{X}^T \underline{\mathbf{D}}_o \mathbf{X}$ and $\mathbf{X}^T \underline{\mathbf{K}}_o \mathbf{X}$ are all diagonal. From this point, it is straightforward to determine the left and right parts of the structure-preserving diagonalising equivalence. These are identical and are given by

$$\underline{\mathbf{T}} = 10^{-3} * \begin{bmatrix} -28.2204 & -21.7725 & 13.8422 & 0.5128 & -0.0109 & -0.0000 \\ -22.4223 & 19.6328 & 20.6002 & 0.2288 & 0.2550 & -0.0000 \\ 16.9498 & -6.5772 & 26.0390 & 0.7345 & 0.4515 & 0.0000 \\ -65.5050 & 5.6367 & -257.0914 & -29.9025 & -21.6542 & 0.0000 \\ -29.2315 & -131.5579 & 76.4689 & -23.1729 & 16.8731 & 24.7175 \\ -93.8257 & -232.9398 & -24.5455 & 14.5405 & -11.4635 & 24.7175 \end{bmatrix}$$

The diagonalised system and its derivatives are calculated from equation (5.121).

Table 5-2 summarises the values of $\{m_i, d_i, k_i\}$ for each of the three pairs of modes as well as $\{\dot{m}_i, \dot{d}_i, \dot{k}_i\}$.

	Mode Pair 1	Mode Pair 2	Mode Pair 3
m_i	7.8254 e-3	1.9378e-3	1.0692e-20
d_i	2.5669e-2	2.0972e-2	5.3763e-2
k_i	9.9964e-1	9.9977e-1	9.9855e-1
\dot{m}_i	0.4139e-2	0.3712e-2	0.672e-2
\dot{d}_i	-0.2159e-2	-0.144e-1	0.1921e-1
\dot{k}_i	0.2304e-4	0.2949e-3	-0.1034e-2

Table 5-2 : Diagonalised system and its rates of change

The rate of change of the diagonalising transformation $\underline{\dot{T}}$ is computed by substituting equations (5.140), (5.141) into equations (5.125), (5.126). Terms involving the unknown scalars, $\{\underline{\beta}_{Li}, \underline{\beta}_{Ri}\}$, vanish naturally because of equations (5.95), (5.96). The derivatives of the equations (5.109)-(5.111) are sufficient to determine $\{\underline{\beta}_{Li}, \underline{\beta}_{Ri}\}$ uniquely. We find

$$\underline{\dot{T}} = 10^{-3} * \begin{bmatrix} 6.3979 & -6.1790 & 16.5810 & 0.3525 & 0.3174 & 1.7303 \\ -4.3913 & 8.4139 & -13.8721 & 0.0949 & 0.2566 & -0.5147 \\ 1.4526 & 22.3316 & 8.5647 & 0.2828 & 0.8830 & 0.1652 \\ -10.3857 & -174.5571 & -1.1403 & 6.2727 & -9.9218 & 11.5605 \\ 3.3407 & 119.5562 & 6.7647 & -4.2423 & 12.8174 & -12.0328 \\ 13.4884 & -9.4664 & 227.8947 & 2.0016 & 25.4895 & 20.3615 \end{bmatrix}$$

5.10.3 Example 3: Undamped non-symmetric system

This example is an undamped system having non-symmetric (2×2) mass and stiffness matrices. It is selected to be an interesting case for conventional eigenvalue solutions because all of the eigenvalues of the system are complex.

$$M_o = \begin{bmatrix} 4 & 3 \\ -5 & 3 \end{bmatrix}, K_o = \begin{bmatrix} 11 & -16 \\ 8 & -2 \end{bmatrix}, M_1 = \begin{bmatrix} 4 & 0 \\ -1 & -4 \end{bmatrix}, K_1 = \begin{bmatrix} -4 & 0 \\ -5 & 4 \end{bmatrix}$$

The eigenvalues comprise two conjugate pairs – one pair being the negative of the other.

$$\lambda = \pm(1.6058 \pm 0.7484i)$$

The decoupled single-degree-of freedom system $\{m_i, d_i, k_i\}$ and their derivatives are calculated. Table 5-3 summarises the values $\{m_i, d_i, k_i\}$ for each of the two pairs of eigenvalues as well as $\{\dot{m}_i, \dot{d}_i, \dot{k}_i\}$.

	Mode Pair 1	Mode Pair 2
m_i	0.2174	0.2174
d_i	0.6981	-0.6981
k_i	0.6822	0.6822
\dot{m}_i	-0.0055	-0.0055
\dot{d}_i	0.0414	-0.0414
\dot{k}_i	-0.0406	-0.0406

Table 5-3 : Diagonalised system and its rates of change

The right and left parts of the diagonalising transformations $\{\underline{T}_R, \underline{T}_L\}$ are given by

$$\underline{T}_R = \begin{bmatrix} -0.0481 & -0.1409 & -0.0601 & 0.0612 \\ -0.2683 & -0.0420 & -0.1478 & -0.0391 \\ 0.1886 & -0.1922 & 0.1449 & 0.0558 \\ 0.4640 & 0.1226 & 0.2065 & -0.1675 \end{bmatrix}, \quad \underline{T}_L = \begin{bmatrix} 0.1118 & 0.1873 & 0.0149 & -0.1240 \\ 0.2426 & -0.2061 & 0.1707 & 0.0444 \\ -0.0468 & 0.3892 & 0.0639 & -0.2109 \\ -0.5357 & -0.1392 & -0.3055 & -0.0636 \end{bmatrix}$$

The rates of change of the right and left diagonalising transformations $\{\dot{\underline{T}}_R, \dot{\underline{T}}_L\}$ are found to be

$$\dot{\underline{T}}_R = \begin{bmatrix} 0.0378 & 0.0268 & 0.0394 & -0.0063 \\ -0.1374 & 0.0333 & -0.0971 & -0.0300 \\ -0.1300 & 0.0264 & -0.0723 & 0.0232 \\ 0.2891 & 0.0901 & 0.2149 & -0.0738 \end{bmatrix}, \quad \dot{\underline{T}}_L = \begin{bmatrix} -0.0647 & -0.0938 & -0.0020 & 0.0518 \\ 0.3657 & -0.0133 & 0.1673 & -0.0178 \\ 0.0080 & -0.1758 & -0.0622 & 0.0387 \\ -0.5068 & 0.0605 & -0.2181 & -0.0583 \end{bmatrix}$$

5.10.4 Example 4: A Physical system

In this example we try to give a practical application of the proposed approach.

Figure 5-2 shows a system with 3 degrees of freedom whose system matrices are

$$\mathbf{M}_o = \begin{bmatrix} \theta_{m1} & 0 & 0 \\ 0 & \theta_{m2} & 0 \\ 0 & 0 & \theta_{m3} \end{bmatrix}, \quad \mathbf{K}_o = \begin{bmatrix} \theta_{k1} & -\theta_{k1} & 0 \\ -\theta_{k1} & (\theta_{k1} + \theta_{k2}) & -\theta_{k2} \\ 0 & -\theta_{k2} & (\theta_{k2} + \theta_{k3}) \end{bmatrix},$$

$$\mathbf{D}_o = \begin{bmatrix} \theta_{d1} & -\theta_{d1} & 0 \\ -\theta_{d1} & (\theta_{d1} + \theta_{d2}) & -\theta_{d2} \\ 0 & -\theta_{d2} & (\theta_{d2} + \theta_{d3}) \end{bmatrix}$$

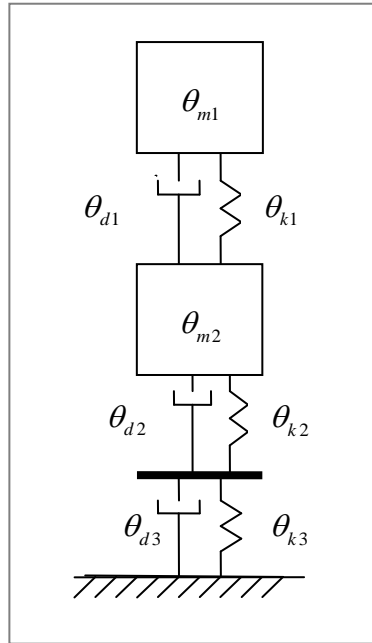


Figure 5-2 : Physical system; mass, stiffness and damping

We will investigate the case where $\theta_{m1} = 2$, $\theta_{m2} = 1$, $\theta_{m3} = 0$, $\theta_{k1} = 3.0e4$, $\theta_{k2} = 2.0e4$, $\theta_{k3} = 1.0e4$ and $\theta_{d1} = 278.9616508149336$, $\theta_{d2} = \theta_{d3} = 100$. The value for θ_{d1} has been chosen so as to cause a pair of identical real roots. The rate of change of the system parameters are $\dot{\theta}_{m1} = 3$, $\dot{\theta}_{m2} = 1$, $\dot{\theta}_{m3} = 5$, $\dot{\theta}_{k1} = 1600$, $\dot{\theta}_{k2} = 1400$, $\dot{\theta}_{k3} = 2000$ and $\dot{\theta}_{d1} = 6$, $\dot{\theta}_{d2} = 7$, $\dot{\theta}_{d3} = 2$.

The eigenvalues of the system comprise one pair of complex conjugate, one pair of identical real roots, one other finite real root and one infinite eigenvalue. The inverses of the system eigenvalues are

$$\lambda^{-1} = \begin{bmatrix} -0.00418 \pm 0.02177i \\ -0.00423 \pm 0.00000i \\ -0.00734 \\ -0.00000 \end{bmatrix}$$

The diagonalised system and its derivatives are calculated from equation (5.121).

Table 5-4 summarises values of $\{m_i, d_i, k_i\}$ for each of the three pairs of modes as well as $\{\dot{m}_i, \dot{d}_i, \dot{k}_i\}$.

	Mode Pair 1	Mode Pair 2	Mode Pair 3
m_i	4.9128e-4	1.8481e-5	0
d_i	8.3581e-3	8.5976e-3	7.3425e-3
k_i	9.9996e-1	9.9993e-1	9.9997e-1
\dot{m}_i	0.1134e-2	0.1278e-3	0.1836e-3
\dot{d}_i	-0.6694e-2	0.1207e-1	-0.1407e-1
\dot{k}_i	0.5539e-6	-0.1038e-3	0.1033e-3

Table 5-4 : Diagonalised system and its rates of change

5.11 Conclusions

This chapter has investigated methods to evaluate the discrepancies between different sets of modal data. The new approaches have generalised the ideas of eigenvalue and eigenvalue derivatives for undamped systems to the concept of structure-preserving equivalences SPEs for generally damped systems. This chapter also has provided a method to calculate the eigenpairs and their derivatives from the

diagonalised system, diagonalising transformations and their derivatives for generally damped system.

The new construction for these derivatives has several advantages over the conventional approaches to eigenvalue and eigenvector derivatives. Firstly, cases where the existence of a pair of identical real roots causes the derivatives of two eigenvalues and their corresponding eigenvectors to become undefined present no such problem in this case. Secondly, cases of infinite eigenvalues (corresponding to some zero singular values in the mass matrix) produce no difficulty whatsoever. Thirdly, some other cases where the Jordan form for the system is non-diagonal can have well defined derivatives for their diagonalised systems and diagonalising transformations.

As a by-product of the development of these new formulae for the derivatives, a new *homogeneous coordinates* expression for pairs of roots of a quadratic eigenvalue problem has been presented (equations (5.95) and (5.96)). This expression potentially has substantial value in its own right and it forms obvious prototypes for related expressions for groups of l eigenvalues of general matrix polynomials of order l .

In some cases the leading coefficients for eigenvalue problems are singular. It might not be possible to find diagonalising SPEs transformations directly for the system. The next chapter addresses new methods to calculate the diagonalising SPEs for ill-conditioned systems.

CHAPTER 6. Spectral Transformations (STs)

This chapter provides new transformations to calculate the diagonalising SPEs where the mass matrix is singular. This case was considered as an example in the previous chapter. Conventional methods for solving the eigenvalue problems of the second order systems are well established for cases where the leading coefficient is non-singular. In these conventional cases it is possible to create a strong linearization. In cases where the leading coefficient is singular we often have numerical difficulties in finding eigenvalues and eigenvectors. Meerbergen [129] introduces an application of matrix transformations for computing large-scale eigenvalue problems. It might not be possible to find diagonalising SPEs transformations directly for the system.

Coordinate transformations of second order systems have been paid substantial attention so far in the literature. The coordinate transformations have returned the eigenvalues and eigenstructure exactly but the eigenvectors have changed. Structure preserving equivalences SPEs are one of transformation that are of interest to us. The SPEs transformations [44] preserve the eigenvalues $\underline{\Lambda}$ but change the modal matrix $\underline{\Phi}$ ($2n \times 2n$). These transformations map between two systems in state space.

Spectral transformation is one of the several transformations which are used to transform system matrices to new system matrices. The spectral transformation maps between two systems having the same eigenvectors but change the eigenvalues. Möbius transformations [40, 130] (or linear fractional transformations) are spectral transformations which map between two systems. These transformations preserve the

modal matrix Φ (sets of second order eigenvectors) ($n \times n$) but change the eigenvalues Λ according to some mapping.

The Möbius transformations include special cases (translation, rotation, linear transformation and inversion) [40]. The benefits of using Möbius transformations are divided into two parts. The first part is to transform the eigenvalue problem of the original second order system with a singular coefficient to new eigenvalue problem system. The second part is to find a relationship between diagonalising SPEs for two systems.

In this chapter we are going to discuss a transformation of an independent scalar variable λ which will change the eigenvalues but not eigenvectors ($n \times n$). This chapter also focuses on calculating the diagonalising SPEs for an ill-conditioned system using the Möbius transformations. There are other methods for calculating the diagonalising SPEs based on Möbius transformations using modal filters and homogenous coordinates (more details in Appendices E and F).

6.1 Eigenvalue problems for general second order systems

The quadratic eigenvalue problem for the system stiffness, damping and mass matrices $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$, respectively, can be represented as

$$(\mathbf{K} + \lambda_i \mathbf{D} + \lambda_i^2 \mathbf{M}) \phi_{Ri} = 0 \quad (6.1)$$

$$\phi_{Li} (\mathbf{K}^T + \lambda_i \mathbf{D}^T + \lambda_i^2 \mathbf{M}^T) = 0 \quad (6.2)$$

In this context, ϕ_{Ri} and ϕ_{Li} define the i^{th} eigenvectors of the right and left short modal matrices $\{\Phi_R, \Phi_L\}$ ($n \times n$) respectively.

Diagonalising SPEs $\{\underline{T}_L, \underline{T}_R\}$ for the system $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ are previously mentioned in Chapter 2 section 2.6 equations (2.42)-(2.44).

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} 0 & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{K}_D \\ \mathbf{K}_D & \mathbf{D}_D \end{bmatrix} \quad (6.3)$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{K} & 0 \\ 0 & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{K}_D & 0 \\ 0 & -\mathbf{M}_D \end{bmatrix} \quad (6.4)$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} -\mathbf{D}_D & -\mathbf{M}_D \\ -\mathbf{M}_D & 0 \end{bmatrix} \quad (6.5)$$

The left and right diagonalising transformations $\{\underline{T}_L, \underline{T}_R\}$ are each divided into four partitions and structures of these blocks are defined in equations (2.81) and (2.82). The i^{th} entry of the diagonalised system $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$ satisfies equation (5.122)

$$k_i + \lambda d_i + \lambda^2 m_i = 0 \quad (6.6)$$

6.2 Möbius Transformations

The general form of Möbius transformations [40] are

$$\lambda = \frac{p\gamma + q}{r\gamma + s} \quad (ps - qr \neq 0) \quad (6.7)$$

The inverse of the Möbius transformations in equation (6.7) are

$$\gamma = \frac{s\lambda - q}{-r\lambda + p} \quad (6.8)$$

A new eigenvalue problem of second order system $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ can be obtained by substituting equation (6.7) into equation (6.1). The new eigenvalue problem system

has the same second order (short) eigenvectors as the original system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ and the eigenvalues related to equation (6.8). Multiplying the result by $(r\gamma + s)^2$ and dividing by $(qr - ps)$

$$\frac{1}{(qr - ps)} \begin{pmatrix} (s^2 \mathbf{K}_o + qs \mathbf{D}_o + q^2 \mathbf{M}_o) + \\ \gamma_i (2rs \mathbf{K}_o + (ps + qr) \mathbf{D}_o + 2pq \mathbf{M}_o) + \\ \gamma_i^2 (r^2 \mathbf{K}_o + pr \mathbf{D}_o + p^2 \mathbf{M}_o) \end{pmatrix} \boldsymbol{\phi}_{Ri} = 0 \quad (6.9)$$

Then, the quadratic eigenvalue problem for the new system can be written as

$$(\mathbf{K}_N + \gamma_i \mathbf{D}_N + \gamma_i^2 \mathbf{M}_N) \boldsymbol{\phi}_{Ri} = 0 \quad (6.10)$$

The left and right short eigenvectors $\{\boldsymbol{\phi}_{Li}, \boldsymbol{\phi}_{Ri}\}$ for the original system and the new system are the same. The system matrices $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ can be expressed in terms of the original system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ matrices as

$$\mathbf{K}_N = s^2 \mathbf{K}_o + qs \mathbf{D}_o + q^2 \mathbf{M}_o \quad (6.11)$$

$$\mathbf{D}_N = 2rs \mathbf{K}_o + (ps + qr) \mathbf{D}_o + 2pq \mathbf{M}_o \quad (6.12)$$

$$\mathbf{M}_N = r^2 \mathbf{K}_o + pr \mathbf{D}_o + p^2 \mathbf{M}_o \quad (6.13)$$

Equations (6.11)-(6.13) define a relationship between the new system and the original system matrices which can be written as

$$\begin{bmatrix} \mathbf{K}_N \\ \mathbf{D}_N \\ \mathbf{M}_N \end{bmatrix} = (\mathbf{Q}_{SSTN} \otimes \mathbf{I}_n) \begin{bmatrix} \mathbf{K}_o \\ \mathbf{D}_o \\ \mathbf{M}_o \end{bmatrix} \quad (6.14)$$

In this framework, the \mathbf{Q}_{SSTN} matrix is called the system spectral transformation matrix (SST) and defined as

$$\mathbf{Q}_{SSTN} = \frac{1}{(qr - ps)} \begin{bmatrix} s^2 & qs & q^2 \\ 2rs & (ps + qr) & 2pq \\ r^2 & rp & p^2 \end{bmatrix} \quad (6.15)$$

and $\{p, q, r, s\}$ are constants, the symbol \otimes represents the Kronecker product and \mathbf{I} represents the $(n \times n)$ identity matrix. Equation (6.13) can be singular, the constants $\{p, q, r, s\}$ can be chosen which can minimise the condition number of \mathbf{M}_N $\min(\text{cond}(\mathbf{M}_N))$ in equation (6.13)

$$\begin{aligned} p &= \cos \theta & q &= \sin \theta \\ r &= \cos \beta & s &= \sin \beta \end{aligned} \quad (6.16)$$

The condition number is the ratio of the largest singular value of \mathbf{M}_N to the smallest. The best value of angles θ and β are chosen at minimum condition of \mathbf{M}_N . Figure 6-1 shows the condition number of \mathbf{M}_N as a function of the angles θ, β for one case.

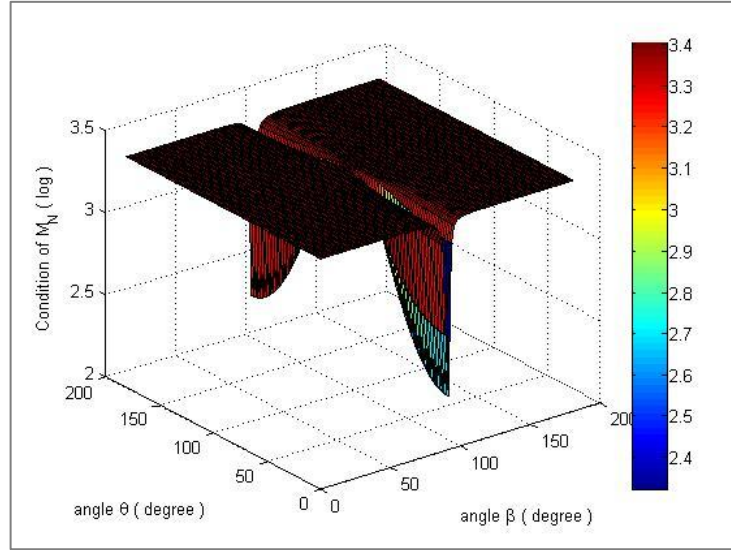


Figure 6-1 : Angles θ , β and condition of \mathbf{M}_N

The eigenvalue problem for the new system can be easily solved using a state space system. The left and right diagonalising transformations $\{\underline{\mathbf{T}}_{LN}, \underline{\mathbf{T}}_{RN}\}$ for the new system satisfy equations (6.3)-(6.5).

Substituting equation (6.8) into equation (6.10) multiplying by $(-r\lambda + p)^2$ and dividing by $(qr - ps)$ gives another relationship between the original system and the new system

$$\begin{bmatrix} \mathbf{K}_o \\ \mathbf{D}_o \\ \mathbf{M}_o \end{bmatrix} = (\mathbf{Q}_{SSTO} \otimes \mathbf{I}_n) \begin{bmatrix} \mathbf{K}_N \\ \mathbf{D}_N \\ \mathbf{M}_N \end{bmatrix} \quad (6.17)$$

where

$$\mathbf{Q}_{SSTO} = \frac{1}{(qr - ps)} \begin{bmatrix} p^2 & -pq & q^2 \\ -2rp & (ps + qr) & -2sq \\ r^2 & -rs & s^2 \end{bmatrix} \quad (6.18)$$

Matrix \mathbf{Q}_{SSTO} is the inverse of the matrix \mathbf{Q}_{SSTN} .

The diagonalised system $\{\mathbf{M}_{Do}, \mathbf{D}_{Do}, \mathbf{K}_{Do}\}$ can be calculated from $\{\mathbf{M}_{DN}, \mathbf{D}_{DN}, \mathbf{K}_{DN}\}$ using the same *SST* matrix \mathbf{Q}_{SSTO} in equation (6.17) or the inverse of *SST* matrix \mathbf{Q}_{SSTN} . Then evidently,

$$\begin{bmatrix} \mathbf{K}_{Do} \\ \mathbf{D}_{Do} \\ \mathbf{M}_{Do} \end{bmatrix} = (\mathbf{Q}_{SSTO} \otimes \mathbf{I}_n) \begin{bmatrix} \mathbf{K}_{DN} \\ \mathbf{D}_{DN} \\ \mathbf{M}_{DN} \end{bmatrix} \quad (6.19)$$

The system spectral transformation matrix $\mathbf{Q}_{SSTN} = \mathbf{Q}_{SSTO}^{-1}$ used to transform the system matrices into other system matrices is the same spectral transformation matrix used to transform the diagonalised system matrices into other diagonalised system matrices. The diagram in Figure 6-2 shows the Möbius (*spectral*) transformations cycle. There are four major steps to calculate the diagonalising structure preserving equivalences (DSPEs) for the original system matrices $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$.

The first step is to transform the original system matrices $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ into new system matrices $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ using the *SST* matrix \mathbf{Q}_{SSTN} as in equations (6.14) and (6.18). The second step is to find the diagonalised system matrices $\{\mathbf{K}_{DN}, \mathbf{D}_{DN}, \mathbf{M}_{DN}\}$ using the diagonalising structure preserving equivalences (DSPEs) for the new system matrices $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$. The third step is to transform the diagonalised system matrices $\{\mathbf{K}_{DN}, \mathbf{D}_{DN}, \mathbf{M}_{DN}\}$ for the new system matrices into diagonalised system matrices $\{\mathbf{K}_{Do}, \mathbf{D}_{Do}, \mathbf{M}_{Do}\}$ for the original system matrices using the *SST* matrix \mathbf{Q}_{SSTO} or inverse *SST* matrix \mathbf{Q}_{SSTN} . The final step is to calculate the diagonalising structure preserving equivalences for the original system matrices $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ and vice versa.

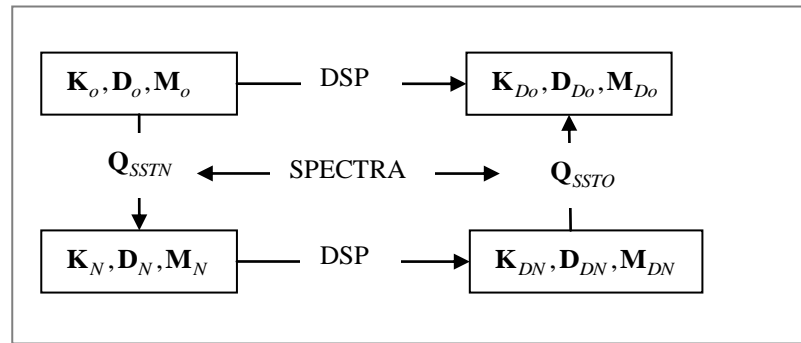


Figure 6-2 : Möbius transformation diagram

The left and right diagonalising SPEs $\{\underline{\mathbf{T}}_{Lo}, \underline{\mathbf{T}}_{Ro}\}$ for the original system matrices $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ can be calculated from the left and right diagonalising SPEs $\{\underline{\mathbf{T}}_{LN}, \underline{\mathbf{T}}_{RN}\}$ for the transformed system matrices $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$. The difficulty here is to find a direct relationship between the diagonalising SPE's for both systems. The next sections can solve this issue.

6.3 Decomposing the Möbius transformations

In this section, we use three different elementary spectral transformations to compose a general Möbius transformation. The three elementary transformations are:

- The shift transformation $\lambda = \alpha + a$
- The invert transformation $\alpha = \frac{1}{\beta}$
- The scaling transformation $\delta = c\gamma$

Both shift and invert transformations are used to convert the largest magnitude of eigenvalues to the nearest shift values that are easy to compute. Once they are found, they may be transformed back to the spectrum of the original system.

Every general Möbius transformation in equation (6.7) can be decomposed into a sequence of elementary transformations [40] as shown in Table 6-1.

Operation Type	Operation	Equation
Shift	$\lambda \rightarrow \alpha$	$\lambda = \alpha + a$
Invert	$\alpha \rightarrow \beta$	$\alpha = \frac{1}{\beta}$
Shift	$\beta \rightarrow \delta$	$\beta = \delta + b$
Scale	$\delta \rightarrow \gamma$	$\delta = c\gamma$

Table 6-1 : Decomposition of the Möbius transformation $\lambda \rightarrow \gamma$

Combining the equations in Table 6-1 yields

$$\lambda = \frac{ac\gamma + (1 + ab)}{c\gamma + b} \quad (6.20)$$

The reverse Möbius transformation can also be decomposed into a sequence of elementary transformations as shown in Table 6-2.

Combining the equations in Table 6-2 yields

$$\gamma = \frac{b\lambda - (1+ab)}{-c\lambda + ac} \quad (6.21)$$

Operation Type	Operation	Equation
Scale	$\gamma \rightarrow \delta$	$\gamma = \frac{\delta}{c}$
Shift	$\delta \rightarrow \beta$	$\delta = \beta - b$
Invert	$\beta \rightarrow \alpha$	$\beta = \frac{1}{\alpha}$
Shift	$\alpha \rightarrow \lambda$	$\alpha = \lambda - a$

Table 6-2 : Decomposition of the Möbius transformation $\gamma \rightarrow \lambda$

6.4 Shift spectral transformation (ST_s)

The shift transformation transforms the original spectrum into a new spectrum directly by using the following equation

$$\lambda = \alpha + a \quad (6.22)$$

Substituting for λ in equation (6.1) leads to

$$\left(\mathbf{K}_o + (\alpha_i + a) \mathbf{D}_o + (\alpha_i + a)^2 \mathbf{M}_o \right) \phi_{Ri} = 0 \quad (6.23)$$

Then, the new system matrices $\{\mathbf{K}_1, \mathbf{D}_1, \mathbf{M}_1\}$ can be represented as

$$\left(\mathbf{K}_1 + \alpha_i \mathbf{D}_1 + \alpha_i^2 \mathbf{M}_1 \right) \phi_{Ri} = 0 \quad (6.24)$$

where $\{\mathbf{K}_1, \mathbf{D}_1, \mathbf{M}_1\}$ are

$$\begin{aligned}
\mathbf{K}_1 &= \mathbf{K}_o + a\mathbf{D}_o + a^2\mathbf{M}_o \\
\mathbf{D}_1 &= \mathbf{D}_o + 2a\mathbf{M}_o \\
\mathbf{M}_1 &= \mathbf{M}_o
\end{aligned} \tag{6.25}$$

Equation (6.25) can be written as

$$\begin{bmatrix} \mathbf{K}_1 \\ \mathbf{D}_1 \\ \mathbf{M}_1 \end{bmatrix} = (\mathbf{Q}_{SST_sa} \otimes \mathbf{I}) \begin{bmatrix} \mathbf{K}_o \\ \mathbf{D}_o \\ \mathbf{M}_o \end{bmatrix} \tag{6.26}$$

The first shift system spectral transformation (SST_sa) matrix \mathbf{Q}_{SST_sa} is

$$\mathbf{Q}_{SST_sa} = \begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & 2a \\ 0 & 0 & 1 \end{bmatrix} \tag{6.27}$$

The diagonalised system matrices $\{\mathbf{K}_{D1}, \mathbf{D}_{D1}, \mathbf{M}_{D1}\}$ for the system $\{\mathbf{K}_1, \mathbf{D}_1, \mathbf{M}_1\}$ can be calculated using the same SST_sa matrix \mathbf{Q}_{SST_sa}

$$\begin{bmatrix} \mathbf{K}_{D1} \\ \mathbf{D}_{D1} \\ \mathbf{M}_{D1} \end{bmatrix} = (\mathbf{Q}_{SST_sa} \otimes \mathbf{I}) \begin{bmatrix} \mathbf{K}_{Do} \\ \mathbf{D}_{Do} \\ \mathbf{M}_{Do} \end{bmatrix} \tag{6.28}$$

The inverse of the matrix \mathbf{Q}_{SST_sa} in equation (6.27) can be used to calculate the diagonalised system $\{\mathbf{K}_{Do}, \mathbf{D}_{Do}, \mathbf{M}_{Do}\}$ for the system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$.

$$\begin{bmatrix} \mathbf{K}_{Do} \\ \mathbf{D}_{Do} \\ \mathbf{M}_{Do} \end{bmatrix} = \left((\mathbf{Q}_{SST_sa})^{-1} \otimes \mathbf{I} \right) \begin{bmatrix} \mathbf{K}_{D1} \\ \mathbf{D}_{D1} \\ \mathbf{M}_{D1} \end{bmatrix} \tag{6.29}$$

where $(\mathbf{Q}_{SST_sa})^{-1}$ is

$$(\mathbf{Q}_{SST_sa})^{-1} = \begin{bmatrix} 1 & -a & a^2 \\ 0 & 1 & -2a \\ 0 & 0 & 1 \end{bmatrix} \tag{6.30}$$

The shift spectral transformation does not change the mass matrix –i.e. $\mathbf{M}_1 = \mathbf{M}_o$.

6.4.1 Shift spectral transformation and diagonalising SPEs

Shifting the spectrum has found a relationship between the left and right diagonalising transformations $\{\underline{\mathbf{T}}_{Lo}, \underline{\mathbf{T}}_{Ro}\}$ for the system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ with the system $\{\mathbf{K}_1, \mathbf{D}_1, \mathbf{M}_1\}$. Pre- and post-multiplying the left and right diagonalising transformations $\{\underline{\mathbf{T}}_{L1}, \underline{\mathbf{T}}_{R1}\}$ for the system $\{\mathbf{K}_1, \mathbf{D}_1, \mathbf{M}_1\}$ by \mathbf{Q}_{DST_sa} and $\mathbf{Q}_{DST_sa}^{-1}$, respectively, yields four partitions in each left and right diagonalising transformation

$$\begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix} = (\mathbf{Q}_{DST_sa} \otimes \mathbf{I})^{-1} \begin{bmatrix} \mathbf{W}_{L1} & \mathbf{X}_{L1} \\ \mathbf{Y}_{L1} & \mathbf{Z}_{L1} \end{bmatrix} (\mathbf{Q}_{DST_sa} \otimes \mathbf{I}) \quad (6.31)$$

$$\begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} = (\mathbf{Q}_{DST_sa} \otimes \mathbf{I})^{-1} \begin{bmatrix} \mathbf{W}_{R1} & \mathbf{X}_{R1} \\ \mathbf{Y}_{R1} & \mathbf{Z}_{R1} \end{bmatrix} (\mathbf{Q}_{DST_sa} \otimes \mathbf{I}) \quad (6.32)$$

where matrix \mathbf{Q}_{DST_sa} is called the shift diagonalising spectral transformation (DST_sa)

$$\mathbf{Q}_{DST_sa} = \begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix} \quad (6.33)$$

6.5 Invert spectral transformation (ST_i)

This transformation inverts the spectrum of the system $\{\mathbf{K}_1, \mathbf{D}_1, \mathbf{M}_1\}$ to another spectrum by using the equation below

$$\alpha = \frac{1}{\beta} \quad (6.34)$$

Substituting for α in equation (6.24) and multiplying by β^2 yields

$$(\beta_i^2 \mathbf{K}_1 + \beta_i \mathbf{D}_1 + \mathbf{M}_1) \phi_{Ri} = 0 \quad (6.35)$$

Then, the system matrices $\{\mathbf{K}_2, \mathbf{D}_2, \mathbf{M}_2\}$ can be represented as

$$(\mathbf{K}_2 + \beta_i \mathbf{D}_2 + \beta_i^2 \mathbf{M}_2) \phi_{Ri} = 0 \quad (6.36)$$

The system matrices $\{\mathbf{K}_2, \mathbf{D}_2, \mathbf{M}_2\}$ are found by replacing the mass and stiffness matrices $\{\mathbf{M}_2, \mathbf{K}_2\}$ with $\{\mathbf{K}_1, \mathbf{M}_1\}$, respectively, whereas $\mathbf{D}_2 = \mathbf{D}_1$.

$$\begin{aligned} \mathbf{K}_2 &= \mathbf{M}_1 \\ \mathbf{D}_2 &= \mathbf{D}_1 \\ \mathbf{M}_2 &= \mathbf{K}_1 \end{aligned} \quad (6.37)$$

writing equation (6.37) as

$$\begin{bmatrix} \mathbf{K}_2 \\ \mathbf{D}_2 \\ \mathbf{M}_2 \end{bmatrix} = (\mathbf{Q}_{SST_i} \otimes \mathbf{I}) \begin{bmatrix} \mathbf{K}_1 \\ \mathbf{D}_1 \\ \mathbf{M}_1 \end{bmatrix} \quad (6.38)$$

The invert system spectral transformation (SST_i) matrix \mathbf{Q}_{SST_i} is

$$\mathbf{Q}_{SST_i} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad (6.39)$$

The diagonalised system $\{\mathbf{K}_{D2}, \mathbf{D}_{D2}, \mathbf{M}_{D2}\}$ can be calculated using the same SST_i matrix \mathbf{Q}_{SST_i} in the form

$$\begin{bmatrix} \mathbf{K}_{D2} \\ \mathbf{D}_{D2} \\ \mathbf{M}_{D2} \end{bmatrix} = (\mathbf{Q}_{SST_i} \otimes \mathbf{I}) \begin{bmatrix} \mathbf{K}_{D1} \\ \mathbf{D}_{D1} \\ \mathbf{M}_{D1} \end{bmatrix} \quad (6.40)$$

The invert spectral transformation does not change the damping matrix $\mathbf{D}_2 = \mathbf{D}_1$ whereas the mass and stiffness matrices are switched $\mathbf{M}_2 = \mathbf{K}_1$ and $\mathbf{K}_2 = \mathbf{M}_1$.

6.5.1 *Invert spectral transformation and diagonalising SPEs*

The diagonalising transformations for the system $\{\mathbf{K}_1, \mathbf{D}_1, \mathbf{M}_1\}$ can be found from the diagonalising transformations for $\{\mathbf{K}_2, \mathbf{D}_2, \mathbf{M}_2\}$ by pre- and post-multiplying LAM's matrices left and right diagonalising transformations $\{\underline{\mathbf{T}}_{L2}, \underline{\mathbf{T}}_{R2}\}$ for the system

$\{\mathbf{K}_2, \mathbf{D}_2, \mathbf{M}_2\}$ by $\mathbf{Q}_{DST_i} = \mathbf{Q}_{DST_i}^{-1}$. The matrix \mathbf{Q}_{DST_i} is called invert diagonalising spectral transformation (DST_i)

$$\mathbf{Q}_{DST_i} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (6.41)$$

6.6 Shift spectral transformation (ST_s)

The second shift transformation is

$$\beta = \delta + b \quad (6.42)$$

The relation between systems $\{\mathbf{K}_3, \mathbf{D}_3, \mathbf{M}_3\}$ and $\{\mathbf{K}_2, \mathbf{D}_2, \mathbf{M}_2\}$ is

$$\begin{bmatrix} \mathbf{K}_3 \\ \mathbf{D}_3 \\ \mathbf{M}_3 \end{bmatrix} = (\mathbf{Q}_{SST_sb} \otimes \mathbf{I}) \begin{bmatrix} \mathbf{K}_2 \\ \mathbf{D}_2 \\ \mathbf{M}_2 \end{bmatrix} \quad (6.43)$$

The second shift system spectral transformation (SST_sb) matrix \mathbf{Q}_{SST_sb} is

$$\mathbf{Q}_{SST_sb} = \begin{bmatrix} 1 & b & b^2 \\ 0 & 1 & 2b \\ 0 & 0 & 1 \end{bmatrix} \quad (6.44)$$

The diagonalised system can be found using the same SST_sb matrix \mathbf{Q}_{SST_sb} as

$$\begin{bmatrix} \mathbf{K}_{D3} \\ \mathbf{D}_{D3} \\ \mathbf{M}_{D3} \end{bmatrix} = (\mathbf{Q}_{SST_sb} \otimes \mathbf{I}) \begin{bmatrix} \mathbf{K}_{D2} \\ \mathbf{D}_{D2} \\ \mathbf{M}_{D2} \end{bmatrix} \quad (6.45)$$

The second shift transformation shows that the mass \mathbf{M}_3 matrix becomes the stiffness \mathbf{K}_2 matrix $\mathbf{M}_3 = \mathbf{K}_2$.

6.6.1 Shift spectral transformation and diagonalising SPEs

The left and right diagonalising transformations for the system $\{\mathbf{K}_2, \mathbf{D}_2, \mathbf{M}_2\}$ can be calculated following the same method which has been mentioned in section 6.4.1.

$$\mathbf{Q}_{DST_sb} = \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix} \quad (6.46)$$

where \mathbf{Q}_{DST_sb} is called second shift diagonalising spectral transformation matrix (DST_sb).

6.7 Scale spectral transformation (ST_sc)

Comparing Möbius transformation in equation (6.7) with those in equations (6.22), (6.34) and (6.42) shows the spectrum requires scaling. The eigenvalues of the system $\{\mathbf{K}_3, \mathbf{D}_3, \mathbf{M}_3\}$ can be multiplied by a scalar as in the following form

$$\delta = c\gamma \quad (6.47)$$

Then, the system matrices $\{\mathbf{K}_4, \mathbf{D}_4, \mathbf{M}_4\}$ are written as

$$(\mathbf{K}_4 + \gamma_i \mathbf{D}_4 + \gamma_i^2 \mathbf{M}_4) \phi_{Ri} = 0 \quad (6.48)$$

where

$$\begin{aligned} \mathbf{K}_4 &= 1/c \mathbf{K}_3 \\ \mathbf{D}_4 &= \mathbf{D}_3 \\ \mathbf{M}_4 &= c \mathbf{M}_3 \end{aligned} \quad (6.49)$$

writing equation (6.49) as

$$\begin{bmatrix} \mathbf{K}_4 \\ \mathbf{D}_4 \\ \mathbf{M}_4 \end{bmatrix} = (\mathbf{Q}_{SST_sc} \otimes \mathbf{I}) \begin{bmatrix} \mathbf{K}_3 \\ \mathbf{D}_3 \\ \mathbf{M}_3 \end{bmatrix} \quad (6.50)$$

The scale system spectral transformation (SST_sc) matrix \mathbf{Q}_{SST_sc} is

$$\mathbf{Q}_{SST_sc} = \begin{bmatrix} 1/c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix} \quad (6.51)$$

The diagonalised system matrices can be found by using the same SST_sc matrix

\mathbf{Q}_{SST_sc} as

$$\begin{bmatrix} \mathbf{K}_{D4} \\ \mathbf{D}_{D4} \\ \mathbf{M}_{D4} \end{bmatrix} = (\mathbf{Q}_{SST_sc} \otimes \mathbf{I}) \begin{bmatrix} \mathbf{K}_{D3} \\ \mathbf{D}_{D3} \\ \mathbf{M}_{D3} \end{bmatrix} \quad (6.52)$$

The inverse of the matrix \mathbf{Q}_{SST_sc} in equation (6.51) can be used to calculate the diagonalised system $\{\mathbf{K}_{D3}, \mathbf{D}_{D3}, \mathbf{M}_{D3}\}$ for the system $\{\mathbf{K}_3, \mathbf{D}_3, \mathbf{M}_3\}$.

$$\begin{bmatrix} \mathbf{K}_{D3} \\ \mathbf{D}_{D3} \\ \mathbf{M}_{D3} \end{bmatrix} = \left((\mathbf{Q}_{SST_sc})^{-1} \otimes \mathbf{I} \right) \begin{bmatrix} \mathbf{K}_{D4} \\ \mathbf{D}_{D4} \\ \mathbf{M}_{D4} \end{bmatrix} \quad (6.53)$$

where $(\mathbf{Q}_{SST_sc})^{-1}$ is

$$(\mathbf{Q}_{SST_sc})^{-1} = \begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/c \end{bmatrix} \quad (6.54)$$

The scale spectral transformation does not change the damping matrix $\mathbf{D}_4 = \mathbf{D}_3$ whereas the mass matrix and the stiffness matrix are changed $\mathbf{M}_4 = c\mathbf{M}_3$ and $\mathbf{K}_4 = \mathbf{K}_3/c$.

6.7.1 Scale spectral transformation and diagonalising SPEs

The left and right diagonalising transformations for the system $\{\mathbf{K}_3, \mathbf{D}_3, \mathbf{M}_3\}$ can be calculated by scaling blocks of the left and right diagonalising transformations for the system $\{\mathbf{K}_4, \mathbf{D}_4, \mathbf{M}_4\}$.

$$\mathbf{Q}_{DST_sc} = \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} \quad (6.55)$$

where the matrix \mathbf{Q}_{DST_sc} is called the scale diagonalising spectral transformation (DST_sc).

6.8 Composing spectral transformations

It is straightforward to find a relationship between the original system matrices with any transformed system matrices. Multiplying the spectral transformation matrices \mathbf{Q}_{SST_sa} , \mathbf{Q}_{SST_i} , \mathbf{Q}_{SST_sb} and \mathbf{Q}_{SST_sc} from equations (6.27), (6.39), (6.44) and (6.51) yields

$$\mathbf{Q}_{SST} = \begin{bmatrix} b^2/c & b(1+ab)/c & (1+ab)^2/c \\ 2b & (1+2ab) & 2a(1+ab) \\ c & ac & a^2c \end{bmatrix} \quad (6.56)$$

The original system matrices $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ and their diagonalised system matrices $\{\mathbf{K}_{Do}, \mathbf{D}_{Do}, \mathbf{M}_{Do}\}$ can be expressed in terms of the transformed system matrices $\{\mathbf{K}_4, \mathbf{D}_4, \mathbf{M}_4\}$ and their diagonalised system matrices $\{\mathbf{K}_{D4}, \mathbf{D}_{D4}, \mathbf{M}_{D4}\}$ with the following equations

$$\begin{bmatrix} \mathbf{K}_4 \\ \mathbf{D}_4 \\ \mathbf{M}_4 \end{bmatrix} = (\mathbf{Q}_{SST} \otimes \mathbf{I}) \begin{bmatrix} \mathbf{K}_o \\ \mathbf{D}_o \\ \mathbf{M}_o \end{bmatrix} \quad (6.57)$$

$$\begin{bmatrix} \mathbf{K}_{D4} \\ \mathbf{D}_{D4} \\ \mathbf{M}_{D4} \end{bmatrix} = (\mathbf{Q}_{SST} \otimes \mathbf{I}) \begin{bmatrix} \mathbf{K}_{Do} \\ \mathbf{D}_{Do} \\ \mathbf{M}_{Do} \end{bmatrix} \quad (6.58)$$

It is possible here to find the left and right diagonalising transformations $\{\underline{\mathbf{T}}_{Lo}, \underline{\mathbf{T}}_{Ro}\}$ for the original system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ in terms of diagonalising transformations for any transformed system matrices. Multiplying the diagonalising spectral transformation

matrices \mathbf{Q}_{DST_sc} , \mathbf{Q}_{DST_sb} , \mathbf{Q}_{DST_i} , and \mathbf{Q}_{DST_sa} for each elementary spectral transformation yields

$$\mathbf{Q}_{DST} = \begin{bmatrix} -ac & c \\ (1+ab) & -b \end{bmatrix} \quad (6.59)$$

The left and right diagonalising transformations $\{\underline{\mathbf{T}}_{Lo}, \underline{\mathbf{T}}_{Ro}\}$ for the system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ can be calculated in terms of $\{\underline{\mathbf{T}}_{L4}, \underline{\mathbf{T}}_{R4}\}$ as

$$\begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix} = (\mathbf{Q}_{DST} \otimes \mathbf{I})^{-1} \begin{bmatrix} \mathbf{W}_{L4} & \mathbf{X}_{L4} \\ \mathbf{Y}_{L4} & \mathbf{Z}_{L4} \end{bmatrix} (\mathbf{Q}_{DST} \otimes \mathbf{I}) \quad (6.60)$$

$$\begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} = (\mathbf{Q}_{DST} \otimes \mathbf{I})^{-1} \begin{bmatrix} \mathbf{W}_{R4} & \mathbf{X}_{R4} \\ \mathbf{Y}_{R4} & \mathbf{Z}_{R4} \end{bmatrix} (\mathbf{Q}_{DST} \otimes \mathbf{I}) \quad (6.61)$$

Figure 6-3 shows that the system spectral transformation matrix \mathbf{Q}_{SST} maps between two systems having the same short eigenvectors and their diagonalised matrices. However, the diagonalising spectral/similarity transformation matrix \mathbf{Q}_{DST} maps between two SPE's having identical eigenvalues.

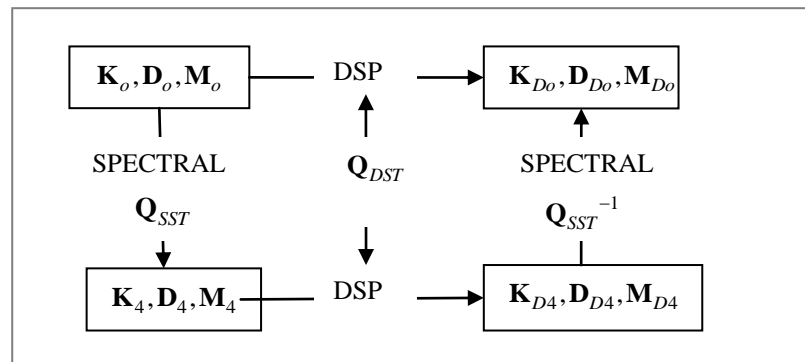


Figure 6-3 : Spectral transformation diagram and SPEs

Comparing equations (6.20) and (6.21) with equations (6.7) and (6.8), it is obvious that

$$\begin{aligned}
p &= ac \\
q &= (1 + ab) \\
r &= c \\
s &= b
\end{aligned} \tag{6.62}$$

The system matrices $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ and their diagonalised matrices $\{\mathbf{K}_{DN}, \mathbf{D}_{DN}, \mathbf{M}_{DN}\}$ can be calculated in terms of new system matrices $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ and their diagonalised matrices $\{\mathbf{K}_{Do}, \mathbf{D}_{Do}, \mathbf{M}_{Do}\}$ using the same spectral transformation matrix \mathbf{Q}_{SSTN} equation (6.15) and vice versa.

$$\begin{bmatrix} \mathbf{K}_N \\ \mathbf{D}_N \\ \mathbf{M}_N \end{bmatrix} = (\mathbf{Q}_{SSTN} \otimes \mathbf{I}) \begin{bmatrix} \mathbf{K}_o \\ \mathbf{D}_o \\ \mathbf{M}_o \end{bmatrix} \tag{6.63}$$

$$\begin{bmatrix} \mathbf{K}_{DN} \\ \mathbf{D}_{DN} \\ \mathbf{M}_{DN} \end{bmatrix} = (\mathbf{Q}_{SSTN} \otimes \mathbf{I}) \begin{bmatrix} \mathbf{K}_{Do} \\ \mathbf{D}_{Do} \\ \mathbf{M}_{Do} \end{bmatrix} \tag{6.64}$$

The left and right diagonalising transformations $\{\underline{\mathbf{T}}_{Lo}, \underline{\mathbf{T}}_{Ro}\}$ for the system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ can be calculated in terms of $\{\underline{\mathbf{T}}_{LN}, \underline{\mathbf{T}}_{RN}\}$ for the system $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ as

$$\begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix} = (\mathbf{Q}_{DSTO} \otimes \mathbf{I})^{-1} \begin{bmatrix} \mathbf{W}_{LN} & \mathbf{X}_{LN} \\ \mathbf{Y}_{LN} & \mathbf{Z}_{LN} \end{bmatrix} (\mathbf{Q}_{DSTO} \otimes \mathbf{I}) \tag{6.65}$$

$$\begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} = (\mathbf{Q}_{DSTO} \otimes \mathbf{I})^{-1} \begin{bmatrix} \mathbf{W}_{RN} & \mathbf{X}_{RN} \\ \mathbf{Y}_{RN} & \mathbf{Z}_{RN} \end{bmatrix} (\mathbf{Q}_{DSTO} \otimes \mathbf{I}) \tag{6.66}$$

where

$$\mathbf{Q}_{DSTO} = \begin{bmatrix} -p & r \\ q & -s \end{bmatrix} \tag{6.67}$$

\mathbf{Q}_{DSTO} and \mathbf{Q}_{DST} are called diagonalising spectral transformation matrices. These transformation matrices are also called similarity transformations and these preserve the eigenvalues [40]. Then, the left and right diagonalising transformations $\{\underline{\mathbf{T}}_{Lo}, \underline{\mathbf{T}}_{Ro}\}$

for the original system matrices $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ are similar to that for the transformed system matrices.

6.9 Example

This example addresses a three degrees of freedom system having a singular mass matrix and repeated real roots. The system matrices $\{\mathbf{K}_0, \mathbf{D}_0, \mathbf{M}_0\}$ are shown below

$$\mathbf{K}_0 = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & (k_1 + k_2) & -k_2 \\ 0 & -k_2 & (k_2 + k_3) \end{bmatrix}, \quad \mathbf{D}_0 = \begin{bmatrix} d_1 & -d_1 & 0 \\ -d_1 & (d_1 + d_2) & -d_2 \\ 0 & -d_2 & (d_2 + d_3) \end{bmatrix}$$

$$\mathbf{M}_0 = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

where $k_1 = 3.0e4$, $k_2 = 2.0e4$, $k_3 = 1.0e4$, $d_1 = 278.9616508149336$, $d_2 = d_3 = 100$ and $m_1 = 2$, $m_2 = 1$, $m_3 = 0$. The value for d_1 has been chosen so as to cause a pair of identical real roots. The eigenvalues of the system comprise one pair of complex conjugates, one pair of identical real roots, one other finite real root and one infinite eigenvalue.

$$\lambda = \begin{bmatrix} -8.5063 - 44.3063i \\ -8.5063 + 44.3063i \\ -232.6196 \\ -232.6196 \\ -136.1906 \\ Inf \end{bmatrix}$$

The left and right eigenvectors $\{\underline{\Phi}_L, \underline{\Phi}_R\}$ for the original system matrices are

$$\underline{\Phi}_L = \underline{\Phi}_R = \begin{bmatrix} -0.00427 + 0.02163i & -0.00427 - 0.02163i & -0.00205 & -0.00205 & 0.00087 & 0.00000 \\ -0.00358 + 0.01875i & -0.00358 - 0.01875i & 0.00429 & 0.00429 & -0.00318 & 0.00000 \\ -0.00331 + 0.01221i & -0.00331 - 0.01221i & 0.00085 & 0.00085 & -0.00734 & -0.00000 \\ 0.99462 + 0.00538i & 0.99462 - 0.00538i & 0.47581 & 0.47581 & -0.11886 & 0.00000 \\ 0.86099 - 0.00087i & 0.86099 + 0.00087i & -1.00000 & -1.00000 & 0.43283 & -0.00000 \\ 0.56908 + 0.04281i & 0.56908 - 0.04281i & -0.19741 & -0.19741 & 1.00000 & -1.00000 \end{bmatrix}$$

The constants $\{p, q, r, s\}$ in equation (6.7) are calculated using equation (6.16). A MATLAB function is used to calculate the best angles θ and β which are chosen at minimum condition of \mathbf{M}_N . Figure 6-4 and Figure 6-5 show the condition of \mathbf{M}_N in logarithm values against the range of angles θ and β in degrees $^\circ$.

$$Best_ \theta = 52.58427, \quad best_ \beta = 89.86854$$

$$cond(\mathbf{M}_N) = 5.55050, \quad \log(cond(\mathbf{M}_N)) = 1.71389$$

The constants p, q, r, s are

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} -0.68017 & 0.73306 \\ -0.32702 & 0.94502 \end{bmatrix}$$

The system spectral transformation matrix \mathbf{Q}_{SSTN} is calculated using equation (6.15)

$$\mathbf{Q}_{SSTN} = \begin{bmatrix} 2.21578 & 1.7188 & 1.3333 \\ -1.5335 & -2.1896 & -2.4742 \\ 0.2653 & 0.5519 & 1.1478 \end{bmatrix}$$

The transformed system matrices $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ are calculated using equations (6.14).

The eigenvalues for the system are

$$\gamma = \begin{bmatrix} 2.8706 \pm 0.0805i \\ 2.8737 \pm 0.0000i \\ 2.8625 \\ 2.8898 \end{bmatrix}$$

The i^{th} vector $\{\phi_{Ri}\}$ of short eigenvector Φ_R and γ_i with the system matrices $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ satisfy equation (6.10).

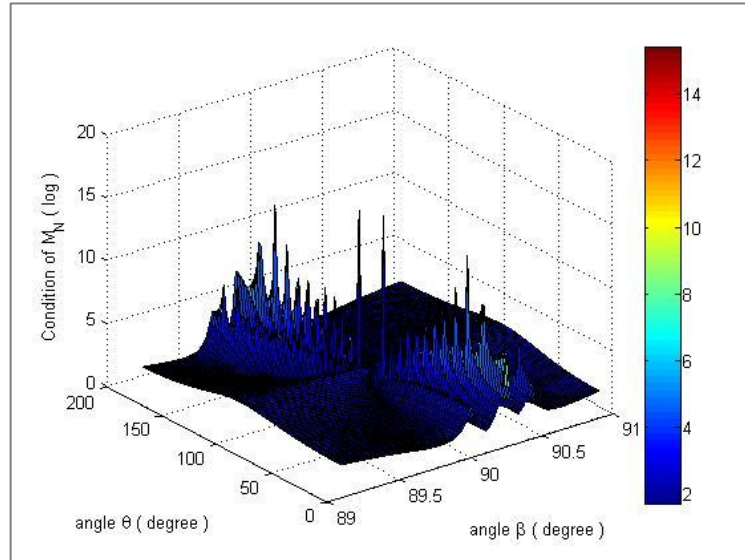


Figure 6-4 : Angles θ and β with condition number of M_N

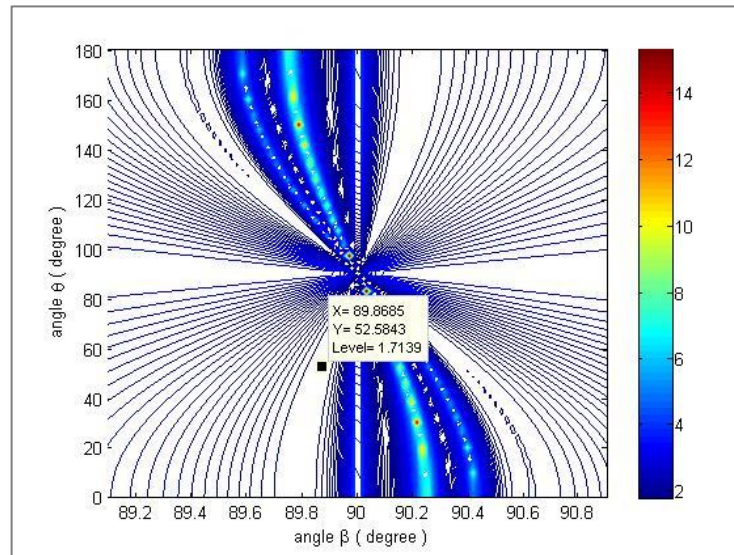


Figure 6-5 : Contours of condition number of M_N

The left and right diagonalising transformations $\{\underline{\mathbf{T}}_{LN}, \underline{\mathbf{T}}_{RN}\}$ for the system $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ and diagonalised system matrices $\{\mathbf{K}_{DN}, \mathbf{D}_{DN}, \mathbf{M}_{DN}\}$ are found to be

$$\mathbf{K}_{DN} = \text{diag} \left(\begin{bmatrix} 0.8167 \\ 0.8168 \\ 0.8169 \end{bmatrix} \right), \quad \mathbf{D}_{DN} = \text{diag} \left(\begin{bmatrix} -0.5685 \\ -0.5684 \\ -0.5681 \end{bmatrix} \right), \quad \mathbf{M}_{DN} = \text{diag} \left(\begin{bmatrix} 0.0990 \\ 0.0989 \\ 0.0988 \end{bmatrix} \right)$$

The left and right diagonalising transformations $\{\underline{\mathbf{T}}_{Lo}, \underline{\mathbf{T}}_{Ro}\}$ for the original system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ are calculated using the diagonalising spectral transformation \mathbf{Q}_{DSTO} in equation (6.67), equations (6.65) and (6.66).

$$\mathbf{Q}_{DSTN} = \begin{bmatrix} 0.6802 & -0.3270 \\ 0.7331 & -0.9450 \end{bmatrix}$$

The diagonalised system matrices $\{\mathbf{K}_{Do}, \mathbf{D}_{Do}, \mathbf{M}_{Do}\}$ are calculated using equations (6.3)-(6.5) and can also be calculated using equation (6.19).

$$\mathbf{K}_{Do} = \text{diag} \left(\begin{bmatrix} 0.3661 \\ 0.3662 \\ 0.3666 \end{bmatrix} \right), \quad \mathbf{D}_{Do} = \text{diag} \left(\begin{bmatrix} 0.0031 \\ 0.0032 \\ 0.0027 \end{bmatrix} \right), \quad \mathbf{M}_{Do} = \text{diag} \left(1.0e-3 \times \begin{bmatrix} 0.1799 \\ 0.0068 \\ 0.0000 \end{bmatrix} \right)$$

6.10 Conclusion

This chapter has investigated methods based on Möbius/spectral transformation formulae to calculate the diagonalising SPEs where the eigenvalue problem leading coefficient is singular. Two new transformations have been investigated called system spectral transformation \mathbf{Q}_{SSTN} and diagonalising spectral/similarity transformation \mathbf{Q}_{DSTO} . The system spectral transformation \mathbf{Q}_{SSTN} maps between matrices of two systems having the same second order (short) eigenvectors and their diagonalised system matrices. The diagonalising spectral transformation \mathbf{Q}_{DSTO} maps between two diagonalising SPE's having identical eigenvalues.

Equation (6.14) shows a relationship between the original system matrices and the transformed system matrices using the Möbius transformation formula in equation (6.7). The matrix \mathbf{Q}_{SSTN} called the system spectral transformation matrix (SST) can also be used to determine the diagonalised system matrices.

The Möbius transformation is decomposed into four elementary spectral transformations. These spectral transformations are used to find a relationship between the diagonalising SPEs for the new system matrices and the original system matrices. The system spectral transformation matrix \mathbf{Q}_{SSTN} maps between two systems and their diagonalised system matrices having the same short eigenvectors. However, the diagonalising spectral/similarity transformation matrix \mathbf{Q}_{DSTO} maps between two SPEs having identical eigenvalues.

The following chapter focuses on generalising modal correlation methods to assess the discrepancies between sets of modal data.

CHAPTER 7. Modal correlation

This chapter focuses on the development of general modal correlation methods to assess the discrepancies between experimental data and analytical data. Modal correlation is well-known as a technique which is performed for comparing measured modes with modes obtained from a numerical analysis. There are many methods for correlating measured modes with analytical modes. The most common and popular techniques known for modal correlation measures are the modal assurance criterion (MAC) and cross orthogonality checks on modal vectors [1-3].

This chapter extends the idea of mass and stiffness-weighted cross orthogonality measures in a natural way to the case of damped systems using *diagonalising structure-preserving equivalences* (SPEs). In this chapter, two different basic classes of orthogonality-*cross orthogonality* and *mutual orthogonality*- are presented. The cross orthogonality measures are used for matching sets of measured modes with sets of analytical modes. These measures are similar to MAC usage.

The *mutual orthogonality* measures are defined through cross orthogonality measures. The *mutual orthogonality* measures are used to evaluate the consistency between measured modal information and system matrices. The *mutual orthogonality* measures can be used for modal updating and do not require matching of modes. Moreover, these calculations can be performed either at the level of the reduced analytical system matrices or by using extended measured modal information.

Every linear second-order system can be represented using the following three equations:

$$\begin{aligned}\mathbf{f} &= \mathbf{S}_L \mathbf{u} \\ \mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} &= \mathbf{f} \\ \mathbf{y} &= \mathbf{S}_R^T \mathbf{q}\end{aligned}\tag{7.1}$$

in which \mathbf{K} , \mathbf{D} and \mathbf{M} are the system stiffness, damping and mass matrices, \mathbf{q} is a vector of (generalised) displacement coordinates, \mathbf{f} is a vector of (generalised) forces, and both of these vectors are functions of time. The matrices are assumed to be constant. The dot notation indicates differentiation with respect to time. The system has n degrees of freedom if \mathbf{q} and \mathbf{f} each contain n entries. Vectors \mathbf{u} and \mathbf{y} represent the model inputs and outputs and they have n_i and n_o entries respectively. The matrix \mathbf{S}_L has dimensions $(n \times n_i)$ and it distributes the system inputs onto the full-length force vector, \mathbf{f} . The matrix \mathbf{S}_R has dimensions $(n \times n_o)$ and it controls how the generalised displacements are manifest in the output vector, \mathbf{y} .

7.1 Undamped system modes

7.1.1 Eigenvalues and eigenvectors

The governing equation of motion for an undamped system is introduced in section 2.3. The eigenvalues $\{\lambda_i, \dots, \lambda_n\}$ with the corresponding right and left eigenvectors $\{\phi_{Ri}, \phi_{Li}\}$ together satisfy the following equations

$$(\mathbf{K} + \lambda_i^2 \mathbf{M}) \phi_{Ri} = 0 \tag{7.2}$$

$$\phi_{Li}^T (\mathbf{K} + \lambda_i^2 \mathbf{M}) = 0 \tag{7.3}$$

where ϕ_{Ri} represents the i^{th} vector of the right modal matrix Φ_R and ϕ_{Li} represents the i^{th} vector of the left modal matrix Φ_L .

The system can be diagonalised by transformations thus

$$\Phi_L^T \mathbf{M} \Phi_R = \mathbf{M}_D \quad (7.4)$$

$$\Phi_L^T \mathbf{K} \Phi_R = \mathbf{K}_D \quad (7.5)$$

where \mathbf{M}_D and \mathbf{K}_D are real-valued diagonal matrices.

7.1.2 *Mass and stiffness weighted orthogonality for undamped systems*

In general modes may not have been mass-normalised. This allows for the possibility that \mathbf{M} need not necessarily be invertible and it also emphasises the important point that the mass-weighted orthogonality of the modes is no more special than the stiffness-weighted orthogonality checks.

There are different possible methods [91-94] which are suggested to achieve the orthogonality requirement. Garvey *et al.* [23] defined the mass and stiffness-weighted orthogonality in the following equations. Equations (2.8) and (7.5) can be written as

$$\Theta^{-1} \Phi_L^T \mathbf{M} \Phi_R \Theta^{-1} = \mathbf{I} \quad (7.6)$$

$$\Psi^{-1} \Phi_L^T \mathbf{K} \Phi_R \Psi^{-1} = \mathbf{I} \quad (7.7)$$

In some cases, the values of \mathbf{M}_D and \mathbf{K}_D are positive then, $\mathbf{M}_D = \Theta^2$ and $\mathbf{K}_D = \Psi^2$.

However, it might be possible that one or more of the entries in \mathbf{M}_D or/and \mathbf{K}_D are

negative. The issue here is that the inverse of Θ and Ψ will have imaginary parts.

The matrices $\{\mathbf{M}_D, \mathbf{K}_D\}$ can be written as

$$\mathbf{M}_D = \Theta_L \Theta_R \quad (7.8)$$

$$\mathbf{K}_D = \Psi_L \Psi_R \quad (7.9)$$

In this case, the values in Θ_L are equal to the values in Θ_R but where a minus sign appears in one, that entry in the other will be positive. The same applies to the $\{\Psi_L, \Psi_R\}$.

7.2 Generally damped modes for second-order systems

7.2.1 Eigenvalues and eigenvectors for damped systems

The governing equation of motion for a damped second order system is introduced in Chapter 2 section 2.1.5. Every linear second-order system can be represented using the following three equations:

The right and left eigenvectors $\{\phi_{Ri}, \phi_{Li}\}$ are also mentioned in Chapter 2 section 2.3.2 which satisfy the following equations

$$(\mathbf{K} + \lambda_i \mathbf{D} + \lambda_i^2 \mathbf{M}) \phi_{Ri} = 0 \quad (7.10)$$

$$\phi_{Li}^T (\mathbf{K} + \lambda_i \mathbf{D} + \lambda_i^2 \mathbf{M}) = 0 \quad (7.11)$$

Equation (7.1) can be written in a state-space form as

$$\begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} - \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} \quad (7.12)$$

when no forcing is present, this becomes

$$\underline{\mathbf{D}}\underline{\dot{\mathbf{x}}} - \underline{\mathbf{K}}\underline{\mathbf{x}} = \mathbf{0} \quad (7.13)$$

where $\underline{\mathbf{D}}$ and $\underline{\mathbf{K}}$ are $(2n \times 2n)$ matrices, $\underline{\mathbf{x}}$ is the so-called *state-vector* and the definitions of all are evident from comparison of equation (2.23) with (7.13).

Coordinate transformations are introduced in Chapter 2 section 2.5. The relationship between two systems is referred as a *conventional* equivalence.

$$\begin{aligned} \mathbf{T}_L^T \mathbf{K} \mathbf{T}_R &= \mathbf{K}_N \\ \mathbf{T}_L^T \mathbf{D} \mathbf{T}_R &= \mathbf{D}_N \\ \mathbf{T}_L^T \mathbf{M} \mathbf{T}_R &= \mathbf{M}_N \end{aligned} \quad (7.14)$$

It is also well known that given any arbitrary system $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$, it is not usually possible to find invertible matrices, $\{\mathbf{T}_L, \mathbf{T}_R\}$ such that $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ are all diagonal.

7.2.2 *Structure preserving equivalences (SPEs)*

The structure preserving transformations (SPTs) which are introduced in Chapter 2, developed by Garvey *et al.* [42, 44] are coordinate transformations for second-order systems. The concept of structure-preserving transformations (SPTs) is described in [47]. These transformations are applied to the *LAMs* such that the structure of *LAMs* is preserved and the eigenvalues of the system are preserved.

Diagonalising SPEs which are introduced in Chapter 2, section 2.6 and Chapter 5, section 5.4 are represented by real-valued $(n \times n)$ matrices $\{\mathbf{W}_L, \mathbf{X}_L, \mathbf{Y}_L, \mathbf{Z}_L\}$, $\{\mathbf{W}_R, \mathbf{X}_R, \mathbf{Y}_R, \mathbf{Z}_R\}$. These transformations $\{\underline{\mathbf{T}}_L, \underline{\mathbf{T}}_R\}$ can usually decouple the equations of motion such that the new system matrices $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$ themselves are diagonal.

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_D \\ \mathbf{K}_D & \mathbf{D}_D \end{bmatrix} \quad (7.15)$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{K}_D & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_D \end{bmatrix} \quad (7.16)$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} -\mathbf{D}_D & -\mathbf{M}_D \\ -\mathbf{M}_D & \mathbf{0} \end{bmatrix} \quad (7.17)$$

More concisely

$$\underline{\mathbf{T}}_L^T \underline{\mathbf{M}} \underline{\mathbf{T}}_R = \underline{\mathbf{M}}_D \quad (7.18)$$

$$\underline{\mathbf{T}}_L^T \underline{\mathbf{D}} \underline{\mathbf{T}}_R = \underline{\mathbf{D}}_D \quad (7.19)$$

$$\underline{\mathbf{T}}_L^T \underline{\mathbf{K}} \underline{\mathbf{T}}_R = \underline{\mathbf{K}}_D \quad (7.20)$$

where $\{\underline{\mathbf{K}}_D, \underline{\mathbf{D}}_D, \underline{\mathbf{M}}_D\}$ are block diagonal matrices and $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$ represent the *LAMs* for the system whose coefficient matrices are the diagonal matrices $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$

$$\begin{aligned} \mathbf{K}_D &= \text{diag}(k_1, k_2, \dots, k_n) \\ \mathbf{D}_D &= \text{diag}(d_1, d_2, \dots, d_n) \\ \mathbf{M}_D &= \text{diag}(m_1, m_2, \dots, m_n) \end{aligned} \quad (7.21)$$

Pairs of columns of the right and left diagonalising transformations $\{\underline{\mathbf{t}}_{Ri}, \underline{\mathbf{t}}_{Li}\}$ satisfy the following equations

$$\left(m_i \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} + d_i \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} + k_i \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \right) \begin{bmatrix} w_{Ri} & x_{Ri} \\ y_{Ri} & z_{Ri} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (7.22)$$

$$\begin{bmatrix} w_{Li} & x_{Li} \\ y_{Li} & z_{Li} \end{bmatrix}^T \left(m_i \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} + d_i \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} + k_i \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (7.23)$$

where it is trivial to verify

$$\left(m_i \begin{bmatrix} 0 & k_i \\ k_i & d_i \end{bmatrix} + d_i \begin{bmatrix} k_i & 0 \\ 0 & -m_i \end{bmatrix} + k_i \begin{bmatrix} -d_i & -m_i \\ -m_i & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (7.24)$$

Equations (7.22) and (7.23) can be written more concisely

$$(m_i \underline{\mathbf{M}} + d_i \underline{\mathbf{D}} + k_i \underline{\mathbf{K}}) \underline{\mathbf{t}}_{Ri} = 0 \quad (7.25)$$

$$\underline{\mathbf{t}}_{Li}^T (m_i \underline{\mathbf{M}} + d_i \underline{\mathbf{D}} + k_i \underline{\mathbf{K}}) = 0 \quad (7.26)$$

7.3 Modal correlation for undamped systems with complete modal information

There are many methods for correlating measured modes with analytical modes. The MAC is the most common measure of modal correlation used. The MAC [23] is independent of modal scaling and was defined originally for real valued modes and extended later to complex modes. One of the modal correlation techniques for general undamped systems is the cross orthogonality method.

It is common practice in the vibration literature to *mass-normalise* modes and if this were done, we would have $\mathbf{M}_D = \mathbf{I}$. However we will retain the generality that the modes may not have been mass-normalised. This emphasises the important point that the mass-weighting orthogonality of the modes is no more special than the stiffness-weighted orthogonality checks.

7.3.1 Orthogonality properties of undamped systems

This section extends the ideas proposed by Garvey *et al.* [23] for undamped symmetric systems. Now consider that there have been two slightly-different undamped systems, $\{\mathbf{K}_o, \mathbf{M}_o\}$ and $\{\mathbf{K}_a, \mathbf{M}_a\}$ - each with slightly different modal properties. The subscripts “o” and “a” refer to the original and analytical systems. Then

$$\Phi_{Lo}^T \mathbf{M}_o \Phi_{Ro} = \mathbf{M}_{Do} \quad (7.27)$$

$$\Phi_{Lo}^T \mathbf{K}_o \Phi_{Ro} = \mathbf{K}_{Do} \quad (7.28)$$

$$\Phi_{La}^T \mathbf{M}_a \Phi_{Ra} = \mathbf{M}_{Da} \quad (7.29)$$

$$\Phi_{La}^T \mathbf{K}_a \Phi_{Ra} = \mathbf{K}_{Da} \quad (7.30)$$

The eigenvalues and corresponding right and left eigenvectors of the standard equations of motion in structural dynamics in second-order form for undamped systems are given by the solutions of equations (5.6) and (5.7). The modal matrices Φ_L, Φ_R satisfy equations (2.8) and (7.5) for measured and analytical systems.

The diagonalising transformations satisfy these equations

$$\Theta_{Lo}^{-1} \Phi_{Lo}^T \mathbf{M}_o \Phi_{Ro} \Theta_{Ro}^{-1} = \mathbf{I}_n \quad (7.31)$$

$$\Theta_{La}^{-1} \Phi_{La}^T \mathbf{M}_a \Phi_{Ra} \Theta_{Ra}^{-1} = \mathbf{I}_n \quad (7.32)$$

$$\Psi_{Lo}^{-1} \Phi_{Lo}^T \mathbf{K}_o \Phi_{Ro} \Psi_{Ro}^{-1} = \mathbf{I}_n \quad (7.33)$$

$$\Psi_{La}^{-1} \Phi_{La}^T \mathbf{K}_a \Phi_{Ra} \Psi_{Ra}^{-1} = \mathbf{I}_n \quad (7.34)$$

in which $\{\Theta_{Lo}, \Theta_{Ro}, \Theta_{La}, \Theta_{Ra}\}$ are some real invertible diagonal matrices such that

$$\Theta_{Lo} \Theta_{Ro} = \mathbf{M}_{Do} \quad (7.35)$$

$$\Theta_{La} \Theta_{Ra} = \mathbf{M}_{Da} \quad (7.36)$$

The generality afforded by allowing $\Theta_{Lo} \neq \Theta_{Ro}$ and $\Theta_{La} \neq \Theta_{Ra}$ allows that \mathbf{M}_{Do} and \mathbf{M}_{Da} may contain negative diagonal entries without requiring that any of $\{\Theta_{Lo}, \Theta_{Ro}, \Theta_{La}, \Theta_{Ra}\}$ should contain complex (purely-imaginary) quantities.

Similarly, the matrices $\{\Psi_{Lo}, \Psi_{Ro}, \Psi_{La}, \Psi_{Ra}\}$ are some real invertible diagonal matrices such that

$$\Psi_{Lo} \Psi_{Ro} = \mathbf{K}_{Do} \quad (7.37)$$

$$\Psi_{La} \Psi_{Ra} = \mathbf{K}_{Da} \quad (7.38)$$

7.3.2 *Mass and stiffness-weighted cross-orthogonality for undamped systems*

Pseudo or cross orthogonality checks have been used in much of the literature to overcome some of the problems associated with some methods of modal correlation. The two mass-weighting orthogonality measures using only the known mass matrix, \mathbf{M}_a are defined as

$$\mathbf{XO}_{Moa} = \mathbf{\Theta}_{Lo}^{-1} \mathbf{\Phi}_{Lo}^T \mathbf{M}_a \mathbf{\Phi}_{Ra} \mathbf{\Theta}_{Ra}^{-1} \quad (7.39)$$

$$\mathbf{XO}_{Mao} = \mathbf{\Theta}_{La}^{-1} \mathbf{\Phi}_{La}^T \mathbf{M}_a \mathbf{\Phi}_{Ro} \mathbf{\Theta}_{Ro}^{-1} \quad (7.40)$$

The matrices \mathbf{XO}_{Moa} and \mathbf{XO}_{Mao} have dimensions $(m \times n)$ and $(n \times m)$, respectively, where m is the number of measured modes and n is the number of degrees of freedom.

There are two different stiffness-weighting *cross-orthogonality* measures - using only \mathbf{K}_a which is also known

$$\mathbf{XO}_{Koa} = \mathbf{\Psi}_{Lo}^{-1} \mathbf{\Phi}_{Lo}^T \mathbf{K}_a \mathbf{\Phi}_{Ra} \mathbf{\Psi}_{Ra}^{-1} \quad (7.41)$$

$$\mathbf{XO}_{Kao} = \mathbf{\Psi}_{La}^{-1} \mathbf{\Phi}_{La}^T \mathbf{K}_a \mathbf{\Phi}_{Ro} \mathbf{\Psi}_{Ro}^{-1} \quad (7.42)$$

By looking at equations (7.39) and (7.41), the *cross orthogonality* measures can help to assess which eigenvectors from the left measured system best matches the right analytical system. In addition, the *cross orthogonality* measures can help to match up which eigenvectors from the right measured system best match with one from the left analytical system by looking at equations (7.40) and (7.42).

7.3.3 Mass and Stiffness mutual orthogonality for undamped system

The products of equations (7.39), (7.40) and equations (7.41), (7.42) in equations (7.43), (7.45) define other measure \mathbf{O}_{Moo} and \mathbf{O}_{Koo} respectively. These measures are independent of any modal information from system $\{\mathbf{K}_a, \mathbf{M}_a\}$. By rearranging equation (7.32), (7.34) and substituting into equation (7.43) and (7.45), respectively, the results will be referred to as mass and stiffness *mutual orthogonality* measures, respectively.

7.3.3.1 Errors in the stiffness matrix

If some discrepancy $\Delta \mathbf{K}$, exists between the physical model and the analytical model in the stiffness matrix then, $(\mathbf{XO}_{Moa} \mathbf{XO}_{Mao})$ must be a bi-orthogonal matrix pair. Matrices $\{\mathbf{XO}_{Moa}, \mathbf{XO}_{Mao}\}$ are said to be bi-orthogonal if $\mathbf{XO}_{Moa} \mathbf{XO}_{Mao} = \mathbf{C}$. \mathbf{C} is the identity matrix and has dimensions $(m \times m)$, where $m < n$. The modal correlation here is the product of the two mass-weighted orthogonality measures equations (7.39) and (7.40).

$$\mathbf{XO}_{Ma} = (\Theta_{Lo}^{-1} \Phi_{Lo}^T \mathbf{M}_a \Phi_{Ra} \Theta_{Ra}^{-1}) \times (\Theta_{La}^{-1} \Phi_{La}^T \mathbf{M}_a \Phi_{Ro} \Theta_{Ro}^{-1}) \quad (7.43)$$

Also equation (7.43) defines a new expression which is called mutual orthogonality measures \mathbf{O}_{Moo} . If the two mass matrices $\{\mathbf{M}_o, \mathbf{M}_a\}$ are identical, then $\mathbf{O}_{Moo} = \mathbf{I}$ which is similar to equation (7.31).

$$\begin{aligned} \mathbf{XO}_{Ma} &= (\Theta_{Lo}^{-1} \Phi_{Lo}^T) \times (\Theta_{La}^{-1} \Phi_{La}^T)^{-1} \times (\Theta_{La}^{-1} \Phi_{La}^T) \times \mathbf{M}_a \times (\Phi_{Ro} \Theta_{Ro}^{-1}) \\ &= (\Theta_{Lo}^{-1} \Phi_{Lo}^T) \times \mathbf{M}_a \times (\Phi_{Ro} \Theta_{Ro}^{-1}) \\ &=: \mathbf{O}_{Moo} \end{aligned} \quad (7.44)$$

7.3.3.2 Errors in the mass matrix

If the physical model differs from the analytical model only by some errors in the mass matrix $\Delta \mathbf{M}$, then $(\mathbf{XO}_{Koa} \mathbf{XO}_{Kao})$ must be a bi-orthogonal matrix pair. The modal correlation here is the product of these two stiffness-weighted orthogonality measures (7.41) and (7.42).

$$\mathbf{XO}_{Ka} = (\Psi_{Lo}^{-1} \Phi_{Lo}^T \mathbf{K}_a \Phi_{Ra} \Psi_{Ra}^{-1}) \times (\Psi_{La}^{-1} \Phi_{La}^T \mathbf{K}_a \Phi_{Ro} \Psi_{Ro}^{-1}) \quad (7.45)$$

Equation (7.45) defines a new expression which is called mutual orthogonality measures \mathbf{O}_{Koo} . If the two stiffness matrices $\{\mathbf{K}_o, \mathbf{K}_a\}$ are identical, then $\mathbf{O}_{Koo} = \mathbf{I}$ which is similar to equation (7.33).

$$\begin{aligned} \mathbf{XO}_{Ka} &= (\Psi_{Lo}^{-1} \Phi_{Lo}^T) \times (\Psi_{La}^T \Phi_{La}^T)^{-1} \times (\Psi_{La}^{-1} \Phi_{La}^T) \times \mathbf{K}_a \times (\Phi_{Ro} \Psi_{Ro}^{-1}) \\ &= (\Psi_{Lo}^{-1} \Phi_{Lo}^T) \times \mathbf{K}_a \times (\Phi_{Ro} \Psi_{Ro}^{-1}) \\ &=: \mathbf{O}_{Koo} \end{aligned} \quad (7.46)$$

Equations (7.43) and (7.45) define the *cross-orthogonality* measures $\mathbf{XO}_{Ma}, \mathbf{XO}_{Ka}$ for undamped systems. These equations require modal information from both systems $\{\mathbf{K}_o, \mathbf{M}_o\}$ and $\{\mathbf{K}_a, \mathbf{M}_a\}$. However, equations (7.44) and (7.46) define the *mutual orthogonality* measures $\mathbf{O}_{Moo}, \mathbf{O}_{Koo}$ which are independent of any information from the system $\{\mathbf{K}_a, \mathbf{M}_a\}$. The measured modal information for the system $\{\mathbf{K}_o, \mathbf{M}_o\}$ are bi-orthogonal through the mass and stiffness matrices.

7.3.4 *Rate of change of cross orthogonality measures \mathbf{o}_{Moo} and \mathbf{o}_{Koo} for undamped systems*

For the purposes of this work, the system matrices $\{\mathbf{K}_a(\theta), \mathbf{M}_a(\theta)\}$ are taken to be linear functions of a scalar parameter vector θ . In effect, we are assuming that stiffness and mass matrices are changing continuously as described in the following equations

$$\mathbf{M}_a(\theta) = \mathbf{M}_o + \sum_{j=1}^p \theta_j \mathbf{M}_j \quad (7.47)$$

$$\mathbf{K}_a(\theta) = \mathbf{K}_o + \sum_{j=1}^p \theta_j \mathbf{K}_j \quad (7.48)$$

Obviously, the eigenvalues and their corresponding eigenvectors are also changing and are functions of the scalar parameter vector θ . Differentiating equations (7.29) and (7.30) with respect to each scalar parameter θ_j in vector θ gives

$$\dot{\mathbf{M}}_{Da} = \dot{\Phi}_{La}^T \mathbf{M}_a \Phi_{Ra} + \Phi_{La}^T \dot{\mathbf{M}}_a \Phi_{Ra} + \Phi_{La}^T \mathbf{M}_a \dot{\Phi}_{Ra} \quad (7.49)$$

$$\dot{\mathbf{K}}_{Da} = \dot{\Phi}_{La}^T \mathbf{K}_a \Phi_{Ra} + \Phi_{La}^T \dot{\mathbf{K}}_a \Phi_{Ra} + \Phi_{La}^T \mathbf{K}_a \dot{\Phi}_{Ra} \quad (7.50)$$

The derivatives of eigenvalues and their corresponding eigenvectors for undamped systems are well known. These derivatives with respect to a single scalar parameter θ_j at $\theta_j = 0$ are presented in Chapter 5. The single scalar parameter θ_j is chosen to be a variable parameter from the uncertainty parameter vector θ .

The cross orthogonality measures are also changed.

$$\mathbf{XO}_{Ma} = \mathbf{XO}_M + \Delta \mathbf{XO}_M = \mathbf{XO}_M + \sum_{j=1}^p \frac{\partial \mathbf{XO}_{Ma}}{\partial \theta_j} \Delta \theta_j \quad (7.51)$$

$$\mathbf{XO}_{Ka} = \mathbf{XO}_K + \Delta \mathbf{XO}_K = \mathbf{XO}_K + \sum_{j=1}^p \frac{\partial \mathbf{XO}_{Ka}}{\partial \theta_j} \Delta \theta_j \quad (7.52)$$

where $j=1:p$, and p is number of uncertainty parameter vector $\boldsymbol{\theta}$.

Computing the sensitivity of cross orthogonality measures can be done either numerically or analytically.

7.3.4.1 Rates of change of cross-orthogonality measures for undamped systems

The rates of change of *different cross-orthogonality* measures can be determined by differentiating equations (7.43) and (7.45) with respect to each scalar parameter θ_j in the vector $\boldsymbol{\theta}$ at $\theta_j = 0$. In this case the modal information for both systems is required.

$$\mathbf{X}\dot{\mathbf{O}}_{Ma} = \left(\boldsymbol{\Phi}_{Lo}^{-1} \boldsymbol{\Phi}_{Lo}^T \begin{pmatrix} \dot{\mathbf{M}}_a \boldsymbol{\Phi}_{Ra} \boldsymbol{\Theta}_{Ra}^{-1} + \\ \mathbf{M}_a \dot{\boldsymbol{\Phi}}_{Ra} \boldsymbol{\Theta}_{Ra}^{-1} + \\ \mathbf{M}_a \boldsymbol{\Phi}_{Ra} \dot{\boldsymbol{\Theta}}_{Ra}^{-1} \end{pmatrix} \right) \left(\begin{pmatrix} \dot{\boldsymbol{\Theta}}_{La}^{-1} \boldsymbol{\Phi}_{La}^T \mathbf{M}_a + \\ \boldsymbol{\Theta}_{La}^{-1} \dot{\boldsymbol{\Phi}}_{La}^T \mathbf{M}_a + \\ \boldsymbol{\Theta}_{La}^{-1} \boldsymbol{\Phi}_{La}^T \dot{\mathbf{M}}_a \end{pmatrix} \boldsymbol{\Phi}_{Ro} \boldsymbol{\Theta}_{Ro}^{-1} \right) \quad (7.53)$$

$$\mathbf{X}\dot{\mathbf{O}}_{Ka} = \left(\boldsymbol{\Psi}_{Lo}^{-1} \boldsymbol{\Phi}_{Lo}^T \begin{pmatrix} \dot{\mathbf{K}}_a \boldsymbol{\Phi}_{Ra} \boldsymbol{\Psi}_{Ra}^{-1} + \\ \mathbf{K}_a \dot{\boldsymbol{\Phi}}_{Ra} \boldsymbol{\Psi}_{Ra}^{-1} + \\ \mathbf{K}_a \boldsymbol{\Phi}_{Ra} \dot{\boldsymbol{\Psi}}_{Ra}^{-1} \end{pmatrix} \right) \left(\begin{pmatrix} \dot{\boldsymbol{\Psi}}_{La}^{-1} \boldsymbol{\Phi}_{La}^T \mathbf{K}_a + \\ \boldsymbol{\Psi}_{La}^{-1} \dot{\boldsymbol{\Phi}}_{La}^T \mathbf{K}_a + \\ \boldsymbol{\Psi}_{La}^{-1} \boldsymbol{\Phi}_{La}^T \dot{\mathbf{K}}_a \end{pmatrix} \boldsymbol{\Phi}_{Ro} \boldsymbol{\Psi}_{Ro}^{-1} \right) \quad (7.54)$$

The rates of change of cross orthogonality in equations (7.53) and (7.54) demand modal information from the system $\{\mathbf{K}_a, \mathbf{M}_a\}$.

7.3.4.2 *Rate of change of mutual orthogonality measures for undamped systems*

The rates of change of *mutual orthogonality* can be determined by differentiating equations (7.44) and (7.46) with respect to each scalar parameter θ_j in the vector $\boldsymbol{\theta}$ at $\theta_j = 0$.

$$\dot{\mathbf{O}}_{Moo} = \boldsymbol{\Theta}_{Lo}^{-1} \boldsymbol{\Phi}_{Lo}^T \dot{\mathbf{M}}_a \boldsymbol{\Phi}_{Ro} \boldsymbol{\Theta}_{Ro}^{-1} \quad (7.55)$$

$$\dot{\mathbf{O}}_{Koo} = \boldsymbol{\Psi}_{Lo}^{-1} \boldsymbol{\Phi}_{Lo}^T \dot{\mathbf{K}}_a \boldsymbol{\Phi}_{Ro} \boldsymbol{\Psi}_{Ro}^{-1} \quad (7.56)$$

The rate of change of cross orthogonality in equations (7.55) and (7.56) are independent of any information from the system $\{\mathbf{K}_a, \mathbf{M}_a\}$.

7.3.5 *Residual measures based on mutual orthogonality for undamped systems*

The above mutual orthogonality measures have long been developed for undamped systems [91, 131, 132]. The residuals $(\mathbf{O}_{Koo} - \mathbf{I}_n)$ and $(\mathbf{O}_{Moo} - \mathbf{I}_n)$ are measures of the difference between systems $\{\mathbf{K}_o, \mathbf{M}_o\}$ and $\{\mathbf{K}_a, \mathbf{M}_a\}$ without loss of generality provided that the system matrices $\{\mathbf{K}_o, \mathbf{M}_o\}$ are invertible. Evidently if both of these measures are zero, then the two systems are identical. Note that, the availability of the full modal information for the system $\{\mathbf{K}_o, \mathbf{M}_o\}$ is assumed.

One criticism of the above two measures in equations (7.44) and (7.46) is that they rely on the invertibility of $\{\mathbf{M}_{Do}, \mathbf{K}_{Do}\}$. A measure which does not demand this invertibility is introduced.

Pre- and post-multiplying equation (7.35) by $\{\Psi_{Lo}, \Psi_{Ro}\}$ and pre- and post-multiplying equation (7.37) by $\{\Theta_{Lo}, \Theta_{Ro}\}$ yields

$$\Psi_{Lo} \mathbf{M}_{Do} \Psi_{Ro} = \Psi_{Lo} \Theta_{Lo} \Theta_{Ro} \Psi_{Ro} =: \mathbf{A} \quad (7.57)$$

$$\Theta_{Lo} \mathbf{K}_{Do} \Theta_{Ro} = \Theta_{Lo} \Psi_{Lo} \Psi_{Ro} \Theta_{Ro} =: \mathbf{A} \quad (7.58)$$

$\{\Psi_{Lo}, \Psi_{Ro}\}$ and $\{\Theta_{Lo}, \Theta_{Ro}\}$ are diagonal matrices, equations (7.57) and (7.58) are identical.

Then, new residuals can be calculated. Pre- and post-multiplying equation (7.44) by $\{(\Psi_{Lo} \Theta_{Lo}), (\Theta_{Ro} \Psi_{Ro})\}$ and equation (7.46) by $\{(\Theta_{Lo} \Psi_{Lo}), (\Psi_{Ro} \Theta_{Ro})\}$ respectively yield

$$\mathbf{R}_{Moo} = \mathbf{O}_M - \Psi_{Lo} \Theta_{Lo} \Theta_{Ro} \Psi_{Ro} = \mathbf{O}_M - \mathbf{M}_{Do} \mathbf{K}_{Do} = \mathbf{O}_M - \mathbf{A} \quad (7.59)$$

$$\mathbf{R}_{Koo} = \mathbf{O}_K - \Theta_{Lo} \Psi_{Lo} \Psi_{Ro} \Theta_{Ro} = \mathbf{O}_K - \mathbf{M}_{Do} \mathbf{K}_{Do} = \mathbf{O}_K - \mathbf{A} \quad (7.60)$$

in which

$$\mathbf{O}_M = \Psi_{Lo} (\Theta_{Lo} \mathbf{O}_{Moo} \Theta_{Ro}) \Psi_{Ro} = \Psi_{Lo} (\Phi_{Lo}^T \mathbf{M}_a \Phi_{Ro}) \Psi_{Ro} \quad (7.61)$$

$$\mathbf{O}_K = \Theta_{Lo} (\Psi_{Lo} \mathbf{O}_{Koo} \Psi_{Ro}) \Theta_{Ro} = \Theta_{Lo} (\Phi_{Lo}^T \mathbf{K}_a \Phi_{Ro}) \Theta_{Ro} \quad (7.62)$$

The matrices \mathbf{O}_{Moo} , and \mathbf{O}_{Koo} are the same as in equations (7.44) and (7.46) respectively.

It is appropriate to vectorise \mathbf{R}_{Moo} , and \mathbf{R}_{Koo} and combine them to define the total residual where the dimension is $(2m^2 \times 1)$.

$$\mathbf{r} = \begin{bmatrix} \text{vec}(\mathbf{R}_{Moo}) \\ \text{vec}(\mathbf{R}_{Koo}) \end{bmatrix} \quad (7.63)$$

The rate of change of equations (7.61) and (7.62) can be written as

$$\dot{\mathbf{O}}_M = \Psi_{Lo} (\Phi_{Lo}^T \dot{\mathbf{M}}_a \Phi_{Ro}) \Psi_{Ro} \quad (7.64)$$

$$\dot{\mathbf{O}}_K = \mathbf{\Phi}_{Lo} \left(\mathbf{\Phi}_{Lo}^T \dot{\mathbf{K}}_a \mathbf{\Phi}_{Ro} \right) \mathbf{\Phi}_{Ro} \quad (7.65)$$

These derivatives are independent of any information from the system $\{\mathbf{K}_a, \mathbf{M}_a\}$.

7.4 Modal correlation for undamped systems with incomplete modal information

In practice, complete measured modal information is not available for any system. Measurements are restricted by the input and output matrices, $\{\mathbf{S}_L, \mathbf{S}_R\}$, which determine where excitation may be applied and where responses may be measured. Measurements are also restricted by the sampling rates and resolution of equipment with the result that the number of modes obtained is restricted - reducing it from n to m , where $m < n$.

In general, we will have incomplete information about the modal properties of the system $\{\mathbf{K}_o, \mathbf{M}_o\}$ with modal properties $\left\{ \mathbf{K}_{Do}, \mathbf{M}_{Do}, \left(\mathbf{S}_L^T \mathbf{\Phi}_{Lo} \right), \left(\mathbf{S}_R^T \mathbf{\Phi}_{Ro} \right) \right\}$ where $\{\mathbf{K}_{Do}, \mathbf{M}_{Do}\}$ are each diagonal ($m \times m$) matrices with $m < n$. Note that matrices $\{\mathbf{S}_L, \mathbf{S}_R\}$ are invertible and identical for the two systems $\{\mathbf{K}_o, \mathbf{M}_o\}$ and $\{\mathbf{K}_a, \mathbf{M}_a\}$ and we consider that these are not subject to any uncertainty.

If $\{\mathbf{S}_L, \mathbf{S}_R\}$ are not invertible, then it is not possible to discover $\{\mathbf{\Phi}_{Lo}, \mathbf{\Phi}_{Ro}\}$ from measurements. In fact what the measurements obtain is $\left\{ \left(\mathbf{S}_L^T \mathbf{\Phi}_L \right), \left(\mathbf{S}_R^T \mathbf{\Phi}_R \right) \right\}$. As stated earlier, if the system matrices are all symmetric, then with appropriate scaling

$\Phi_{Lo} = \Phi_{Ro}$ and this can be used to expand the obtained information to some extent (provided that $S_L \neq S_R$). However, this is not useful in the general case.

With modal information obtained from only a subset of the total set of degrees of freedom, the literature is divided between some papers which advocate expanding the modal vectors, $\{\Phi_{Lo}, \Phi_{Ro}\}$ [133, 134], and some [132] which prefer to reduce the system matrices, $\{K_a, M_a\}$. To expand the modal vectors, some transformation matrices, $\{T_L, T_R\}$, are chosen such that

$$S_L^T T_L = I \quad (7.66)$$

$$S_R^T T_R = I \quad (7.67)$$

Then the expanded modal matrices are found as

$$\tilde{\Phi}_{Lo} = T_L (S_L^T \Phi_{Lo}) \quad (7.68)$$

$$\tilde{\Phi}_{Ro} = T_R (S_R^T \Phi_{Ro}) \quad (7.69)$$

Equations (7.68) and (7.69) preclude the possibility that different transformation matrices $\{T_L, T_R\}$ might be developed for each individual pair of eigenvectors such as [133] describes. Including this generality is possible but it is beyond the scope of the present chapter. By accepting the restriction indicated by equations (7.68) and (7.69), it is obvious that the expanded modal matrices will be consistent with the measured modal information in so far as

$$S_L^T \tilde{\Phi}_{Lo} = S_L^T \Phi_{Lo} \quad (7.70)$$

$$S_R^T \tilde{\Phi}_{Ro} = S_R^T \Phi_{Ro} \quad (7.71)$$

The alternative to expanding the modal matrices – namely reducing the system matrices $\{\mathbf{K}_a, \mathbf{M}_a\}$ – is achieved by forming

$$\mathbf{M}_{a_redu} = \mathbf{T}_L^T \mathbf{M}_a \mathbf{T}_R \quad (7.72)$$

$$\mathbf{K}_{a_redu} = \mathbf{T}_L^T \mathbf{K}_a \mathbf{T}_R \quad (7.73)$$

It is clear from equations (7.70) and (7.71) that whether the expanded modal information or the original modal information is employed, the same mutual-orthogonality measures $\{\mathbf{O}_{Koo}, \mathbf{O}_{Moo}\}$ are obtained since

$$\Theta_L^{-1} (\mathbf{S}_L^T \Phi_{Lo})^T (\mathbf{T}_L^T \mathbf{M}_a \mathbf{T}_R) (\mathbf{S}_R^T \Phi_{Ro}) \Theta_R^{-1} = \Theta_L^{-1} (\mathbf{S}_L^T \tilde{\Phi}_{Lo})^T (\mathbf{T}_L^T \mathbf{M}_a \mathbf{T}_R) (\mathbf{S}_R^T \tilde{\Phi}_{Ro}) \Theta_R^{-1} \quad (7.74)$$

$$\Psi_L^{-1} (\mathbf{S}_L^T \Phi_{Lo})^T (\mathbf{T}_L^T \mathbf{K}_a \mathbf{T}_R) (\mathbf{S}_R^T \Phi_{Ro}) \Psi_R^{-1} = \Psi_L^{-1} (\mathbf{S}_L^T \tilde{\Phi}_{Lo})^T (\mathbf{T}_L^T \mathbf{K}_a \mathbf{T}_R) (\mathbf{S}_R^T \tilde{\Phi}_{Ro}) \Psi_R^{-1} \quad (7.75)$$

The same transformation matrices are applicable whether we are expanding the measured modal information or reducing the dimensions of the system matrices.

7.5 Modal Correlation measures for general second order systems

In this section, the general ideas for modal correlation of undamped systems are extended to the case of damped systems. Garvey *et al.* [23] addressed modal correlation measures for general viscous damped structures. This section extends the ideas well presented in that paper.

7.5.1 Orthogonality properties of damped systems

The orthogonality measures for a damped system can be written as

$$\begin{bmatrix} \Psi_L & (\Pi\Theta_L) \\ \mathbf{0} & \Psi_L \end{bmatrix}^{-T} \begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} \begin{bmatrix} \Psi_R & (\Pi\Theta_R) \\ \mathbf{0} & \Psi_R \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \quad (7.76)$$

$$2 \begin{bmatrix} \Psi_L & \Theta_L \\ \Psi_L & -\Theta_L \end{bmatrix}^{-T} \begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} \begin{bmatrix} \Psi_R & \Theta_R \\ \Psi_R & -\Theta_R \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \quad (7.77)$$

$$\begin{bmatrix} -\Theta_L & \mathbf{0} \\ (\Pi\Psi_L) & \Theta_L \end{bmatrix}^{-T} \begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} \begin{bmatrix} -\Theta_R & \mathbf{0} \\ (\Pi\Psi_R) & \Theta_R \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \quad (7.78)$$

The first task here is to generalise the idea of factorising the matrices of the diagonalised system in a way which is equivalent to equations (7.35)-(7.38). This generalisation takes the form:

$$\begin{bmatrix} \Psi_L & (\Pi\Theta_L) \\ \mathbf{0} & \Psi_L \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Psi_R & (\Pi\Theta_R) \\ \mathbf{0} & \Psi_R \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_D \\ \mathbf{K}_D & \mathbf{D}_D \end{bmatrix} \quad (7.79)$$

$$\frac{1}{2} \begin{bmatrix} \Psi_L & \Theta_L \\ \Psi_L & -\Theta_L \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Psi_R & \Theta_R \\ \Psi_R & -\Theta_R \end{bmatrix} = \begin{bmatrix} \mathbf{K}_D & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_D \end{bmatrix} \quad (7.80)$$

$$\begin{bmatrix} -\Theta_L & \mathbf{0} \\ (\Pi\Psi_L) & \Theta_L \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} -\Theta_R & \mathbf{0} \\ (\Pi\Psi_R) & \Theta_R \end{bmatrix} = \begin{bmatrix} -\mathbf{D}_D & -\mathbf{M}_D \\ -\mathbf{M}_D & \mathbf{0} \end{bmatrix} \quad (7.81)$$

The second subscripts (*o* or *a*) have been omitted from equations (7.79)-(7.81) since they are taken to apply to systems $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ and $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$. Observe that if equations (7.79)-(7.81) are expanded, the following equations are obtained

$$\mathbf{M}_D = \Theta_L \Theta_R \quad (7.82)$$

$$\mathbf{K}_D = \Psi_L \Psi_R \quad (7.83)$$

(which are consistent with equations (7.35),(7.36) and (7.37),(7.38) respectively) along with

$$\mathbf{D}_D = \Pi(\Theta_L \Psi_R + \Psi_L \Theta_R) \quad (7.84)$$

Again, in equations (7.82)-(7.84), the second subscripts (o or a) have been omitted and these equations are intended to be applied again to systems $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ and $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$. Note that

$$\begin{bmatrix} \Psi_L & (\Pi \Theta_L) \\ \mathbf{0} & \Psi_L \end{bmatrix}^{-T} \begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} \begin{bmatrix} \Psi_R & (\Pi \Theta_R) \\ \mathbf{0} & \Psi_R \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \quad (7.85)$$

$$2 \begin{bmatrix} \Psi_L & \Theta_L \\ \Psi_L & -\Theta_L \end{bmatrix}^{-T} \begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} \begin{bmatrix} \Psi_R & \Theta_R \\ \Psi_R & -\Theta_R \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \quad (7.86)$$

$$\begin{bmatrix} -\Theta_L & \mathbf{0} \\ (\Pi \Psi_L) & \Theta_L \end{bmatrix}^{-T} \begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} \begin{bmatrix} -\Theta_R & \mathbf{0} \\ (\Pi \Psi_R) & \Theta_R \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \quad (7.87)$$

Equations (7.85), (7.86) and (7.87) can be written as

$$\underline{\mathbf{M}}_{LC}^{-T} \underline{\mathbf{T}}_L^T \underline{\mathbf{M}} \underline{\mathbf{T}}_R \underline{\mathbf{M}}_{RC}^{-1} = \underline{\mathbf{J}} \quad (7.88)$$

$$2 \underline{\mathbf{D}}_{LC}^{-T} \underline{\mathbf{T}}_L^T \underline{\mathbf{D}} \underline{\mathbf{T}}_R \underline{\mathbf{D}}_{RC}^{-1} = \underline{\mathbf{J}} \quad (7.89)$$

$$\underline{\mathbf{K}}_{LC}^{-T} \underline{\mathbf{T}}_L^T \underline{\mathbf{K}} \underline{\mathbf{T}}_R \underline{\mathbf{K}}_{RC}^{-1} = \underline{\mathbf{J}} \quad (7.90)$$

and these equations apply both to $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ and $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$.

7.5.2 Cross orthogonality measures for general damped systems

Three different *cross orthogonality* measures of modal correlation are expressed in terms of the structure preserving equivalences in [23]. The two different cross-orthogonality measures-using the known matrix $\underline{\mathbf{M}}_a$ are defined as

$$\underline{\mathbf{XO}}_{Moa} = \begin{bmatrix} \Psi_{Lo} & (\Pi_o \Theta_{Lo}) \\ \mathbf{0} & \Psi_{Lo} \end{bmatrix}^{-T} \begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K}_a \\ \mathbf{K}_a & \mathbf{D}_a \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ra} & \mathbf{X}_{Ra} \\ \mathbf{Y}_{Ra} & \mathbf{Z}_{Ra} \end{bmatrix} \begin{bmatrix} \Psi_{Ra} & (\Pi_a \Theta_{Ra}) \\ \mathbf{0} & \Psi_{Ra} \end{bmatrix}^{-1} \quad (7.91)$$

$$\underline{\mathbf{XO}}_{Mao} = \begin{bmatrix} \Psi_{La} & (\Pi_a \Theta_{La}) \\ \mathbf{0} & \Psi_{La} \end{bmatrix}^{-T} \begin{bmatrix} \mathbf{W}_{La} & \mathbf{X}_{La} \\ \mathbf{Y}_{La} & \mathbf{Z}_{La} \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K}_a \\ \mathbf{K}_a & \mathbf{D}_a \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} \begin{bmatrix} \Psi_{Ro} & (\Pi_o \Theta_{Ro}) \\ \mathbf{0} & \Psi_{Ro} \end{bmatrix}^{-1} \quad (7.92)$$

More concisely

$$\underline{\mathbf{XO}}_{Moa} = \underline{\mathbf{M}}_{LCo}^{-T} \underline{\mathbf{T}}_{Lo}^T \underline{\mathbf{M}}_a \underline{\mathbf{T}}_{Ra} \underline{\mathbf{M}}_{RCa}^{-1} \quad (7.93)$$

$$\underline{\mathbf{XO}}_{Mao} = \underline{\mathbf{M}}_{LCo}^{-T} \underline{\mathbf{T}}_{La}^T \underline{\mathbf{M}}_a \underline{\mathbf{T}}_{Ro} \underline{\mathbf{M}}_{RCo}^{-1} \quad (7.94)$$

Correspondingly, define two different *cross orthogonality* measures-using the known

$\underline{\mathbf{D}}_a$ matrix

$$\underline{\mathbf{XO}}_{Doa} = 2 \begin{bmatrix} \underline{\Psi}_{Lo} & \underline{\Theta}_{Lo} \\ \underline{\Psi}_{Lo} & -\underline{\Theta}_{Lo} \end{bmatrix}^{-T} \begin{bmatrix} \underline{\mathbf{W}}_{Lo} & \underline{\mathbf{X}}_{Lo} \\ \underline{\mathbf{Y}}_{Lo} & \underline{\mathbf{Z}}_{Lo} \end{bmatrix}^T \begin{bmatrix} \underline{\mathbf{K}}_a & \mathbf{0} \\ \mathbf{0} & -\underline{\mathbf{M}}_a \end{bmatrix} \begin{bmatrix} \underline{\mathbf{W}}_{Ra} & \underline{\mathbf{X}}_{Ra} \\ \underline{\mathbf{Y}}_{Ra} & \underline{\mathbf{Z}}_{Ra} \end{bmatrix} \begin{bmatrix} \underline{\Psi}_{Ra} & \underline{\Theta}_{Ra} \\ \underline{\Psi}_{Ra} & -\underline{\Theta}_{Ra} \end{bmatrix}^{-1} \quad (7.95)$$

$$\underline{\mathbf{XO}}_{Dao} = 2 \begin{bmatrix} \underline{\Psi}_{La} & \underline{\Theta}_{La} \\ \underline{\Psi}_{La} & -\underline{\Theta}_{La} \end{bmatrix}^{-T} \begin{bmatrix} \underline{\mathbf{W}}_{La} & \underline{\mathbf{X}}_{La} \\ \underline{\mathbf{Y}}_{La} & \underline{\mathbf{Z}}_{La} \end{bmatrix}^T \begin{bmatrix} \underline{\mathbf{K}}_a & \mathbf{0} \\ \mathbf{0} & -\underline{\mathbf{M}}_a \end{bmatrix} \begin{bmatrix} \underline{\mathbf{W}}_{Ro} & \underline{\mathbf{X}}_{Ro} \\ \underline{\mathbf{Y}}_{Ro} & \underline{\mathbf{Z}}_{Ro} \end{bmatrix} \begin{bmatrix} \underline{\Psi}_{Ro} & \underline{\Theta}_{Ro} \\ \underline{\Psi}_{Ro} & -\underline{\Theta}_{Ro} \end{bmatrix}^{-1} \quad (7.96)$$

More concisely

$$\underline{\mathbf{XO}}_{Doa} = 2 \underline{\mathbf{D}}_{LCo}^{-T} \underline{\mathbf{T}}_{Lo}^T \underline{\mathbf{D}}_a \underline{\mathbf{T}}_{Ra} \underline{\mathbf{D}}_{RCa}^{-1} \quad (7.97)$$

$$\underline{\mathbf{XO}}_{Dao} = 2 \underline{\mathbf{D}}_{LCo}^{-T} \underline{\mathbf{T}}_{La}^T \underline{\mathbf{D}}_a \underline{\mathbf{T}}_{Ro} \underline{\mathbf{D}}_{RCo}^{-1} \quad (7.98)$$

Define another two different cross-orthogonality measures-using $\underline{\mathbf{K}}_a$ which is also

known

$$\underline{\mathbf{XO}}_{Koa} = \begin{bmatrix} -\underline{\Theta}_{Lo} & \mathbf{0} \\ (\underline{\Pi}_o \underline{\Psi}_{Lo}) & \underline{\Theta}_{Lo} \end{bmatrix}^{-T} \begin{bmatrix} \underline{\mathbf{W}}_{Lo} & \underline{\mathbf{X}}_{Lo} \\ \underline{\mathbf{Y}}_{Lo} & \underline{\mathbf{Z}}_{Lo} \end{bmatrix}^T \begin{bmatrix} -\underline{\mathbf{D}}_a & -\underline{\mathbf{M}}_a \\ -\underline{\mathbf{M}}_a & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{W}}_{Ra} & \underline{\mathbf{X}}_{Ra} \\ \underline{\mathbf{Y}}_{Ra} & \underline{\mathbf{Z}}_{Ra} \end{bmatrix} \begin{bmatrix} -\underline{\Theta}_{Ra} & \mathbf{0} \\ (\underline{\Pi}_a \underline{\Psi}_{Ra}) & \underline{\Theta}_{Ra} \end{bmatrix}^{-1} \quad (7.99)$$

$$\underline{\mathbf{XO}}_{Kao} = \begin{bmatrix} -\underline{\Theta}_{La} & \mathbf{0} \\ (\underline{\Pi}_a \underline{\Psi}_{La}) & \underline{\Theta}_{La} \end{bmatrix}^{-T} \begin{bmatrix} \underline{\mathbf{W}}_{La} & \underline{\mathbf{X}}_{La} \\ \underline{\mathbf{Y}}_{La} & \underline{\mathbf{Z}}_{La} \end{bmatrix}^T \begin{bmatrix} -\underline{\mathbf{D}}_a & -\underline{\mathbf{M}}_a \\ -\underline{\mathbf{M}}_a & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{W}}_{Ro} & \underline{\mathbf{X}}_{Ro} \\ \underline{\mathbf{Y}}_{Ro} & \underline{\mathbf{Z}}_{Ro} \end{bmatrix} \begin{bmatrix} -\underline{\Theta}_{Ro} & \mathbf{0} \\ (\underline{\Pi}_a \underline{\Psi}_{Ro}) & \underline{\Theta}_{Ro} \end{bmatrix}^{-1} \quad (7.100)$$

More concisely

$$\underline{\mathbf{XO}}_{Koa} = \underline{\mathbf{K}}_{LCo}^{-T} \underline{\mathbf{T}}_{Lo}^T \underline{\mathbf{K}}_a \underline{\mathbf{T}}_{Ra} \underline{\mathbf{K}}_{RCa}^{-1} \quad (7.101)$$

$$\underline{\mathbf{XO}}_{Kao} = \underline{\mathbf{K}}_{LCo}^{-T} \underline{\mathbf{T}}_{La}^T \underline{\mathbf{K}}_a \underline{\mathbf{T}}_{Ro} \underline{\mathbf{K}}_{RCo}^{-1} \quad (7.102)$$

Evidently, the main important point by looking at equations (7.91), (7.95) and (7.99)

is that the *cross orthogonality* measures can help to assess which modal information

from the left measured system best match with the right analytical system. Furthermore, the *cross orthogonality* measures can help to match up which modal information from the right measured system best match with one from the left analytical system by looking at equations (7.92), (7.96) and (7.100).

7.5.3 *Mutual orthogonality for general damped systems*

7.5.3.1 *Errors in the mass matrix*

If the physical model differs from the analytical model only by some errors in the mass matrix $\Delta \mathbf{M}$, then $\left(\underline{\mathbf{XO}}_{Moa} \underline{\mathbf{J}}_{(2n \times 2n)} \underline{\mathbf{XO}}_{Mao} \right)$ must be a bi- $\underline{\mathbf{J}}$ -unitary matrix pair.

Matrices $\left\{ \underline{\mathbf{XO}}_{Moa} \underline{\mathbf{J}}_{(2n \times 2n)} \underline{\mathbf{XO}}_{Mao} \right\}$ are said to be bi- $\underline{\mathbf{J}}$ -unitary matrix if

$$\underline{\mathbf{XO}}_{Moa} \underline{\mathbf{J}}_{(2n \times 2n)} \underline{\mathbf{XO}}_{Mao} = \underline{\mathbf{J}}. \quad \underline{\mathbf{J}} \text{ is a matrix which takes form } \underline{\mathbf{J}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \mathbf{I} \text{ is the identity}$$

matrix and has dimensions $(m \times m)$, where $m < n$. The modal correlation here is the product of equations (7.91) and (7.92).

$$\left(\underline{\mathbf{XO}}_{Moa} \underline{\mathbf{J}}_{(2n \times 2n)} \underline{\mathbf{XO}}_{Mao} \right) = \left(\begin{bmatrix} \underline{\Psi}_{Lo} & (\underline{\Pi}_o \underline{\Theta}_{Lo}) \\ \mathbf{0} & \underline{\Psi}_{Lo} \end{bmatrix}^T \begin{bmatrix} \underline{\mathbf{W}}_{Lo} & \underline{\mathbf{X}}_{Lo} \\ \underline{\mathbf{Y}}_{Lo} & \underline{\mathbf{Z}}_{Lo} \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \underline{\mathbf{K}}_a \\ \underline{\mathbf{K}}_a & \underline{\mathbf{D}}_a \end{bmatrix} \begin{bmatrix} \underline{\mathbf{W}}_{Ra} & \underline{\mathbf{X}}_{Ra} \\ \underline{\mathbf{Y}}_{Ra} & \underline{\mathbf{Z}}_{Ra} \end{bmatrix} \begin{bmatrix} \underline{\Psi}_{Ra} & (\underline{\Pi}_a \underline{\Theta}_{Ra}) \\ \mathbf{0} & \underline{\Psi}_{Ra} \end{bmatrix}^{-1} \underline{\mathbf{J}}_{(2n \times 2n)} \right. \\ \left. \begin{bmatrix} \underline{\Psi}_{La} & (\underline{\Pi}_a \underline{\Theta}_{La}) \\ \mathbf{0} & \underline{\Psi}_{La} \end{bmatrix}^T \begin{bmatrix} \underline{\mathbf{W}}_{La} & \underline{\mathbf{X}}_{La} \\ \underline{\mathbf{Y}}_{La} & \underline{\mathbf{Z}}_{La} \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \underline{\mathbf{K}}_a \\ \underline{\mathbf{K}}_a & \underline{\mathbf{D}}_a \end{bmatrix} \begin{bmatrix} \underline{\mathbf{W}}_{Ro} & \underline{\mathbf{X}}_{Ro} \\ \underline{\mathbf{Y}}_{Ro} & \underline{\mathbf{Z}}_{Ro} \end{bmatrix} \begin{bmatrix} \underline{\Psi}_{Ro} & (\underline{\Pi}_o \underline{\Theta}_{Ro}) \\ \mathbf{0} & \underline{\Psi}_{Ro} \end{bmatrix}^{-1} \right) \quad (7.103)$$

A more concise way of writing equation (7.103) is

$$\underline{\mathbf{XO}}_{Ma} = \left(\underline{\mathbf{M}}_{LCo}^{-T} \underline{\mathbf{T}}_{Lo}^T \underline{\mathbf{M}}_a \underline{\mathbf{T}}_{Ra} \underline{\mathbf{M}}_{RCa}^{-1} \underline{\mathbf{J}}_{(2n \times 2n)} \underline{\mathbf{M}}_{LCa}^{-T} \underline{\mathbf{T}}_{La}^T \underline{\mathbf{M}}_a \underline{\mathbf{T}}_{Ro} \underline{\mathbf{M}}_{RCo}^{-1} \right) \quad (7.104)$$

The product of cross-orthogonality measures in equation (7.103) defines another measure, $\underline{\mathbf{O}}_{Moo}$ which is independent of any modal information from the system

$\{\underline{\mathbf{K}}_a, \underline{\mathbf{D}}_a, \underline{\mathbf{M}}_a\}$. Rearranging equation (7.76) for the system $\{\underline{\mathbf{K}}_a, \underline{\mathbf{D}}_a, \underline{\mathbf{M}}_a\}$ and

substituting into equation (7.103) the result will be referred to as *mutual orthogonality* for the system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$. If the two matrices $\{\underline{\mathbf{M}}_o, \underline{\mathbf{M}}_a\}$ are identical,

$$\underline{\mathbf{O}}_{Moo} = \underline{\mathbf{J}}.$$

$$\underline{\mathbf{XO}}_{Ma} =$$

$$\begin{bmatrix} \underline{\Psi}_{Lo} & (\underline{\Pi}_o \underline{\Theta}_{Lo}) \\ \mathbf{0} & \underline{\Psi}_{Lo} \end{bmatrix}^{-T} \begin{bmatrix} \underline{\mathbf{W}}_{Lo} & \underline{\mathbf{X}}_{Lo} \\ \underline{\mathbf{Y}}_{Lo} & \underline{\mathbf{Z}}_{Lo} \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K}_a \\ \mathbf{K}_a & \mathbf{D}_a \end{bmatrix} \begin{bmatrix} \underline{\mathbf{W}}_{Ro} & \underline{\mathbf{X}}_{Ro} \\ \underline{\mathbf{Y}}_{Ro} & \underline{\mathbf{Z}}_{Ro} \end{bmatrix} \begin{bmatrix} \underline{\Psi}_{Ro} & (\underline{\Pi}_o \underline{\Theta}_{Ro}) \\ \mathbf{0} & \underline{\Psi}_{Ro} \end{bmatrix}^{-1} =: \underline{\mathbf{O}}_{Moo} \quad (7.105)$$

Equation (7.105) can be written as

$$\underline{\mathbf{XO}}_{Ma} = \left(\underline{\mathbf{M}}_{LCo}^{-T} \underline{\mathbf{T}}_{Lo}^T \underline{\mathbf{M}}_a \underline{\mathbf{T}}_{Ro} \underline{\mathbf{M}}_{RCo}^{-1} \right) =: \underline{\mathbf{O}}_{Moo} \quad (7.106)$$

7.5.3.2 Errors in the damping matrix

If the physical model differs from the analytical model only by some errors in the damping matrix $\Delta \mathbf{D}$, then $\left(\underline{\mathbf{XO}}_{Doa} \underline{\mathbf{J}}_{(2n \times 2n)} \underline{\mathbf{XO}}_{Dao} \right)$ must be a bi- $\underline{\mathbf{J}}$ -unitary matrix pair.

The modal correlation here is the product of equations (7.95) and (7.96).

$$\underline{\mathbf{XO}}_{Doa} \underline{\mathbf{J}}_{(2n \times 2n)} \underline{\mathbf{XO}}_{Dao} =$$

$$4 \left(\begin{bmatrix} \underline{\Psi}_{Lo} & \underline{\Theta}_{Lo} \\ \underline{\Psi}_{Lo} & -\underline{\Theta}_{Lo} \end{bmatrix}^{-T} \begin{bmatrix} \underline{\mathbf{W}}_{Lo} & \underline{\mathbf{X}}_{Lo} \\ \underline{\mathbf{Y}}_{Lo} & \underline{\mathbf{Z}}_{Lo} \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_a & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_a \end{bmatrix} \begin{bmatrix} \underline{\mathbf{W}}_{Ra} & \underline{\mathbf{X}}_{Ra} \\ \underline{\mathbf{Y}}_{Ra} & \underline{\mathbf{Z}}_{Ra} \end{bmatrix} \begin{bmatrix} \underline{\Psi}_{Ra} & \underline{\Theta}_{Ra} \\ \underline{\Psi}_{Ra} & -\underline{\Theta}_{Ra} \end{bmatrix}^{-1} \underline{\mathbf{J}}_{(2n \times 2n)} \right. \\ \left. \begin{bmatrix} \underline{\Psi}_{La} & \underline{\Theta}_{La} \\ \underline{\Psi}_{La} & -\underline{\Theta}_{La} \end{bmatrix}^{-T} \begin{bmatrix} \underline{\mathbf{W}}_{La} & \underline{\mathbf{X}}_{La} \\ \underline{\mathbf{Y}}_{La} & \underline{\mathbf{Z}}_{La} \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_a & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_a \end{bmatrix} \begin{bmatrix} \underline{\mathbf{W}}_{Ro} & \underline{\mathbf{X}}_{Ro} \\ \underline{\mathbf{Y}}_{Ro} & \underline{\mathbf{Z}}_{Ro} \end{bmatrix} \begin{bmatrix} \underline{\Psi}_{Ro} & \underline{\Theta}_{Ro} \\ \underline{\Psi}_{Ro} & -\underline{\Theta}_{Ro} \end{bmatrix}^{-1} \right) \quad (7.107)$$

More concisely than equation (7.107)

$$\underline{\mathbf{XO}}_{Da} = 4 \left(\underline{\mathbf{D}}_{LCo}^{-T} \underline{\mathbf{T}}_{Lo}^T \underline{\mathbf{D}}_a \underline{\mathbf{T}}_{Ra} \underline{\mathbf{D}}_{RCa}^{-1} \underline{\mathbf{J}}_{(2n \times 2n)} \underline{\mathbf{D}}_{LCo}^{-T} \underline{\mathbf{T}}_{La}^T \underline{\mathbf{D}}_a \underline{\mathbf{T}}_{Ro} \underline{\mathbf{D}}_{RCo}^{-1} \right) \quad (7.108)$$

The product of cross-orthogonality measures in equation (7.107) defines another measure, $\underline{\mathbf{O}}_{Doo}$ which is independent of any modal information from the system

$\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$. The result of rearranging equation (7.77) for the system $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$

and substituting into equation (7.107) will be referred to as the *mutual orthogonality*

for system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$. If the two matrices $\{\underline{\mathbf{D}}_o, \underline{\mathbf{D}}_a\}$ are identical, $\underline{\mathbf{O}}_{Doo} = \underline{\mathbf{J}}$.

$$\underline{\mathbf{XO}}_{Da} = 2 \begin{bmatrix} \underline{\Psi}_{Lo} & \underline{\Theta}_{Lo} \\ \underline{\Psi}_{Lo} & -\underline{\Theta}_{Lo} \end{bmatrix}^{-T} \begin{bmatrix} \underline{\mathbf{W}}_{Lo} & \underline{\mathbf{X}}_{Lo} \\ \underline{\mathbf{Y}}_{Lo} & \underline{\mathbf{Z}}_{Lo} \end{bmatrix}^T \begin{bmatrix} \underline{\mathbf{K}}_a & \mathbf{0} \\ \mathbf{0} & -\underline{\mathbf{M}}_a \end{bmatrix} \begin{bmatrix} \underline{\mathbf{W}}_{Ro} & \underline{\mathbf{X}}_{Ro} \\ \underline{\mathbf{Y}}_{Ro} & \underline{\mathbf{Z}}_{Ro} \end{bmatrix} \begin{bmatrix} \underline{\Psi}_{Ro} & \underline{\Theta}_{Ro} \\ \underline{\Psi}_{Ro} & -\underline{\Theta}_{Ro} \end{bmatrix}^{-1} =: \underline{\mathbf{O}}_{Doo} \quad (7.109)$$

Equation (7.109) can be written as

$$\underline{\mathbf{XO}}_{Da} = 2 \left(\underline{\mathbf{D}}_{LCo}^{-T} \underline{\mathbf{T}}_{Lo}^T \underline{\mathbf{D}}_a \underline{\mathbf{T}}_{Ro} \underline{\mathbf{D}}_{RCo}^{-1} \right) =: \underline{\mathbf{O}}_{Doo} \quad (7.110)$$

7.5.3.3 Errors in the stiffness matrix

If the physical model differs from the analytical model only by some errors in the

stiffness matrix $\Delta \mathbf{K}$, then $\left(\underline{\mathbf{XO}}_{Koa} \underline{\mathbf{J}}_{(2n \times 2n)} \underline{\mathbf{XO}}_{Kao} \right)$ must be a bi- $\underline{\mathbf{J}}$ -unitary matrix pair.

The modal correlation here is the product of equations (7.99) and (7.100).

$$\underline{\mathbf{XO}}_{Koa} \underline{\mathbf{J}}_{(2n \times 2n)} \underline{\mathbf{XO}}_{Kao} = \left(\begin{bmatrix} -\underline{\Theta}_{Lo} & \mathbf{0} \\ (\underline{\Pi}_o \underline{\Psi}_{Lo}) & \underline{\Theta}_{Lo} \end{bmatrix}^{-T} \begin{bmatrix} \underline{\mathbf{W}}_{Lo} & \underline{\mathbf{X}}_{Lo} \\ \underline{\mathbf{Y}}_{Lo} & \underline{\mathbf{Z}}_{Lo} \end{bmatrix}^T \begin{bmatrix} -\underline{\mathbf{D}}_a & -\underline{\mathbf{M}}_a \\ -\underline{\mathbf{M}}_a & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{W}}_{Ra} & \underline{\mathbf{X}}_{Ra} \\ \underline{\mathbf{Y}}_{Ra} & \underline{\mathbf{Z}}_{Ra} \end{bmatrix} \begin{bmatrix} -\underline{\Theta}_{Ra} & \mathbf{0} \\ (\underline{\Pi}_a \underline{\Psi}_{Ra}) & \underline{\Theta}_{Ra} \end{bmatrix}^{-1} \underline{\mathbf{J}}_{(2n \times 2n)} \right. \\ \left. \begin{bmatrix} -\underline{\Theta}_{La} & \mathbf{0} \\ (\underline{\Pi}_a \underline{\Psi}_{La}) & \underline{\Theta}_{La} \end{bmatrix}^{-T} \begin{bmatrix} \underline{\mathbf{W}}_{La} & \underline{\mathbf{X}}_{La} \\ \underline{\mathbf{Y}}_{La} & \underline{\mathbf{Z}}_{La} \end{bmatrix}^T \begin{bmatrix} -\underline{\mathbf{D}}_a & -\underline{\mathbf{M}}_a \\ -\underline{\mathbf{M}}_a & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{W}}_{Ro} & \underline{\mathbf{X}}_{Ro} \\ \underline{\mathbf{Y}}_{Ro} & \underline{\mathbf{Z}}_{Ro} \end{bmatrix} \begin{bmatrix} -\underline{\Theta}_{Ro} & \mathbf{0} \\ (\underline{\Pi}_o \underline{\Psi}_{Ro}) & \underline{\Theta}_{Ro} \end{bmatrix}^{-1} \right) \quad (7.111)$$

Writing equation (7.111) more concisely

$$\underline{\mathbf{XO}}_{Ka} = \left(\underline{\mathbf{K}}_{LCo}^{-T} \underline{\mathbf{T}}_{Lo}^T \underline{\mathbf{K}}_a \underline{\mathbf{T}}_{Ra} \underline{\mathbf{D}}_{RCa}^{-1} \underline{\mathbf{J}}_{(2n \times 2n)} \underline{\mathbf{K}}_{LCo}^{-T} \underline{\mathbf{T}}_{La}^T \underline{\mathbf{K}}_a \underline{\mathbf{T}}_{Ro} \underline{\mathbf{K}}_{RCo}^{-1} \right) \quad (7.112)$$

The product of cross-orthogonality measures in equation (7.111) also defines another

measure, $\underline{\mathbf{O}}_{Koo}$ which is independent of any modal information from the system

$\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$. The result of rearranging equation (7.78) for the system $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$

and substituting into equation (7.111) will be referred to as the *mutual orthogonality*

for the system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$. If the two matrices $\{\underline{\mathbf{K}}_o, \underline{\mathbf{K}}_a\}$ are identical, $\underline{\mathbf{O}}_{Koo} = \underline{\mathbf{J}}$.

$$\underline{\mathbf{XO}}_{Ka} = \begin{bmatrix} -\underline{\Theta}_{Lo} & \mathbf{0} \\ (\underline{\Pi}_o \underline{\Psi}_{Lo}) & \underline{\Theta}_{Lo} \end{bmatrix}^{-T} \begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix}^T \begin{bmatrix} -\mathbf{D}_a & -\mathbf{M}_a \\ -\mathbf{M}_a & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} \begin{bmatrix} -\underline{\Theta}_{Ro} & \mathbf{0} \\ (\underline{\Pi}_o \underline{\Psi}_{Ro}) & \underline{\Theta}_{Ro} \end{bmatrix}^{-1} =: \underline{\mathbf{O}}_{Koo} \quad (7.113)$$

Equation (7.113) can be written as

$$\underline{\mathbf{XO}}_{Ka} = \left(\underline{\mathbf{K}}_{LCo}^{-T} \underline{\mathbf{T}}_{Lo}^T \underline{\mathbf{K}}_a \underline{\mathbf{T}}_{Ro} \underline{\mathbf{K}}_{RCo}^{-1} \right) =: \underline{\mathbf{O}}_{Koo} \quad (7.114)$$

Equations (7.103), (7.107) and (7.111) define the *cross-orthogonality* measures $\underline{\mathbf{XO}}_{Ma}, \underline{\mathbf{XO}}_{Da}, \underline{\mathbf{XO}}_{Ka}$ for general damped systems. These equations require modal information from both systems $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ and $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$. However, equations (7.105), (7.109) and (7.113) define the *mutual orthogonality* measures $\underline{\mathbf{O}}_{Moo}, \underline{\mathbf{O}}_{Doo}, \underline{\mathbf{O}}_{Koo}$ which are independent of any information from the system $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$. The measured modal information for the system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ are bi- $\underline{\mathbf{J}}$ -unitary through the system matrices.

7.5.4 Rate of change of different cross orthogonality measures $\underline{\mathbf{XO}}_{Ma}$, $\underline{\mathbf{XO}}_{Da}$ and $\underline{\mathbf{XO}}_{Ka}$ for damped systems

A procedure is applied which is a close parallel to the presented procedure in section 7.3.4. In effect, we are assuming that stiffness, damping and mass matrices $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$ are changing continuously as shown in the following equations

$$\mathbf{M}_a(\theta) = \mathbf{M}_o + \sum_{j=1}^p \theta_j \mathbf{M}_j \quad (7.115)$$

$$\mathbf{D}_a(\theta) = \mathbf{D}_o + \sum_{j=1}^p \theta_j \mathbf{D}_j \quad (7.116)$$

$$\mathbf{K}_a(\theta) = \mathbf{K}_o + \sum_{j=1}^p \theta_j \mathbf{K}_j \quad (7.117)$$

The eigenvalues and their corresponding eigenvectors are also changed. Then the derivatives of the diagonalised system and diagonalising SPEs can be calculated. Differentiating equations (7.18), (7.19) and (7.20) with respect to each scalar parameter θ_j in vector $\boldsymbol{\theta}$ gives

$$\dot{\mathbf{T}}_{La}^T \mathbf{M}_a \mathbf{T}_{Ra} + \mathbf{T}_{La}^T \dot{\mathbf{M}}_a \mathbf{T}_{Ra} + \mathbf{T}_{La}^T \mathbf{M}_a \dot{\mathbf{T}}_{Ra} = \dot{\mathbf{M}}_{Da} \quad (7.118)$$

$$\dot{\mathbf{T}}_{La}^T \mathbf{D}_a \mathbf{T}_{Ra} + \mathbf{T}_{La}^T \dot{\mathbf{D}}_a \mathbf{T}_{Ra} + \mathbf{T}_{La}^T \mathbf{D}_a \dot{\mathbf{T}}_{Ra} = \dot{\mathbf{D}}_{Da} \quad (7.119)$$

$$\dot{\mathbf{T}}_{La}^T \mathbf{K}_a \mathbf{T}_{Ra} + \mathbf{T}_{La}^T \dot{\mathbf{K}}_a \mathbf{T}_{Ra} + \mathbf{T}_{La}^T \mathbf{K}_a \dot{\mathbf{T}}_{Ra} = \dot{\mathbf{K}}_{Da} \quad (7.120)$$

The derivatives of the diagonalised system matrices and diagonalising SPEs for damped system in equations (7.115), (7.116) and (7.117) with respect to a single scalar parameter θ_j at $\theta_j=0$ are introduced in Chapter 5 and presented in [121, 135].

The *cross orthogonality* measures are also affected by the changes that occur in the system matrices. Matching the discrepancies between the physical and computed modal information can be achieved.

$$\mathbf{XO}_{Ma} = \mathbf{XO}_M + \Delta \mathbf{XO}_M = \mathbf{XO}_M + \sum_{j=1}^p \frac{\partial \mathbf{XO}_{Ma}}{\partial \theta_j} \Delta \theta_j \quad (7.121)$$

$$\mathbf{XO}_{Da} = \mathbf{XO}_D + \Delta \mathbf{XO}_D = \mathbf{XO}_D + \sum_{j=1}^p \frac{\partial \mathbf{XO}_{Da}}{\partial \theta_j} \Delta \theta_j \quad (7.122)$$

$$\mathbf{XO}_{Ka} = \mathbf{XO}_K + \Delta \mathbf{XO}_K = \mathbf{XO}_K + \sum_{j=1}^p \frac{\partial \mathbf{XO}_{Ka}}{\partial \theta_j} \Delta \theta_j \quad (7.123)$$

The sensitivity of *cross orthogonality* measures can be computed by relating changes due to variations in each scalar parameter θ_j . This can be done either numerically or analytically.

7.5.4.1 Rates of change of cross orthogonality measures for damped systems

The rates of change of different cross-orthogonality measures for structure preserving transformations can be determined by differentiating equations (7.104), (7.108) and (5.44) with respect to each scalar parameter θ_j in the vector $\boldsymbol{\theta}$ at $\theta_j = 0$

$$\underline{\mathbf{X}\dot{\mathbf{O}}_{Ma}} = \left(\begin{array}{c} \underline{\mathbf{M}_{LCa}}^{-T} \underline{\mathbf{T}_{Lo}}^T \left(\begin{array}{c} \underline{\dot{\mathbf{M}}_a} \underline{\mathbf{T}_{Ra}} \underline{\mathbf{M}_{RCa}}^{-1} + \\ \underline{\mathbf{M}_a} \underline{\dot{\mathbf{T}}_{Ra}} \underline{\mathbf{M}_{RCa}}^{-1} + \\ \underline{\mathbf{M}_a} \underline{\mathbf{T}_{Ra}} \underline{\dot{\mathbf{M}}_{RCa}}^{-1} \end{array} \right) \end{array} \right) \underline{\mathbf{J}}_{(2n \times 2n)} \left(\begin{array}{c} \left(\begin{array}{c} \underline{\dot{\mathbf{M}}_{LCa}}^{-T} \underline{\mathbf{T}_{La}}^T \underline{\mathbf{M}_a} + \\ \underline{\mathbf{M}_{LCa}}^{-T} \underline{\dot{\mathbf{T}}_{La}}^T \underline{\mathbf{M}_a} + \\ \underline{\mathbf{M}_{LCa}}^{-T} \underline{\mathbf{T}_{La}}^T \underline{\dot{\mathbf{M}}_a} \end{array} \right) \underline{\mathbf{T}_{Ro}} \underline{\mathbf{M}_{RCo}}^{-1} \end{array} \right) \quad (7.124)$$

$$\underline{\mathbf{X}\dot{\mathbf{O}}_{Da}} = 4 \left(\begin{array}{c} \underline{\mathbf{D}_{LCa}}^{-T} \underline{\mathbf{T}_{Lo}}^T \left(\begin{array}{c} \underline{\dot{\mathbf{D}}_a} \underline{\mathbf{T}_{Ra}} \underline{\mathbf{D}_{RCa}}^{-1} + \\ \underline{\mathbf{D}_a} \underline{\dot{\mathbf{T}}_{Ra}} \underline{\mathbf{D}_{RCa}}^{-1} + \\ \underline{\mathbf{D}_a} \underline{\mathbf{T}_{Ra}} \underline{\dot{\mathbf{D}}_{RCa}}^{-1} \end{array} \right) \end{array} \right) \underline{\mathbf{J}}_{(2n \times 2n)} \left(\begin{array}{c} \left(\begin{array}{c} \underline{\dot{\mathbf{D}}_{LCa}}^{-T} \underline{\mathbf{T}_{La}}^T \underline{\mathbf{D}_a} + \\ \underline{\mathbf{D}_{LCa}}^{-T} \underline{\dot{\mathbf{T}}_{La}}^T \underline{\mathbf{D}_a} + \\ \underline{\mathbf{D}_{LCa}}^{-T} \underline{\mathbf{T}_{La}}^T \underline{\dot{\mathbf{D}}_a} \end{array} \right) \underline{\mathbf{T}_{Ro}} \underline{\mathbf{D}_{RCo}}^{-1} \end{array} \right) \quad (7.125)$$

$$\underline{\mathbf{X}\dot{\mathbf{O}}_{Ka}} = \left(\begin{array}{c} \underline{\mathbf{K}_{LCa}}^{-T} \underline{\mathbf{T}_{Lo}}^T \left(\begin{array}{c} \underline{\dot{\mathbf{K}}_a} \underline{\mathbf{T}_{Ra}} \underline{\mathbf{D}_{RCa}}^{-1} + \\ \underline{\mathbf{K}_a} \underline{\dot{\mathbf{T}}_{Ra}} \underline{\mathbf{D}_{RCa}}^{-1} + \\ \underline{\mathbf{K}_a} \underline{\mathbf{T}_{Ra}} \underline{\dot{\mathbf{D}}_{RCa}}^{-1} \end{array} \right) \end{array} \right) \underline{\mathbf{J}}_{(2n \times 2n)} \left(\begin{array}{c} \left(\begin{array}{c} \underline{\dot{\mathbf{K}}_{LCa}}^{-T} \underline{\mathbf{T}_{La}}^T \underline{\mathbf{K}_a} + \\ \underline{\mathbf{K}_{LCa}}^{-T} \underline{\dot{\mathbf{T}}_{La}}^T \underline{\mathbf{K}_a} + \\ \underline{\mathbf{K}_{LCa}}^{-T} \underline{\mathbf{T}_{La}}^T \underline{\dot{\mathbf{K}}_a} \end{array} \right) \underline{\mathbf{T}_{Ro}} \underline{\mathbf{K}_{RCo}}^{-1} \end{array} \right) \quad (7.126)$$

7.5.4.2 Rates of change of mutual orthogonality measures for damped systems

The rates of change of different mutual orthogonality measures for structure preserving transformations can be determined by differentiating equations (7.105), (7.109) and (7.113) with respect to each scalar parameter θ_j in the vector $\boldsymbol{\theta}$ at $\theta_j = 0$

$$\underline{\dot{\mathbf{O}}_{Moo}} = \begin{bmatrix} \underline{\Psi}_{Lo} & (\underline{\Pi}_o \underline{\Theta}_{Lo}) \\ \mathbf{0} & \underline{\Psi}_{Lo} \end{bmatrix}^{-T} \begin{bmatrix} \underline{\mathbf{W}}_{Lo} & \underline{\mathbf{X}}_{Lo} \\ \underline{\mathbf{Y}}_{Lo} & \underline{\mathbf{Z}}_{Lo} \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \underline{\dot{\mathbf{K}}}_a \\ \underline{\dot{\mathbf{K}}}_a & \underline{\dot{\mathbf{D}}}_a \end{bmatrix} \begin{bmatrix} \underline{\mathbf{W}}_{Ro} & \underline{\mathbf{X}}_{Ro} \\ \underline{\mathbf{Y}}_{Ro} & \underline{\mathbf{Z}}_{Ro} \end{bmatrix} \begin{bmatrix} \underline{\Psi}_{Ro} & (\underline{\Pi}_o \underline{\Theta}_{Ro}) \\ \mathbf{0} & \underline{\Psi}_{Ro} \end{bmatrix}^{-1} \quad (7.127)$$

$$\underline{\dot{\mathbf{O}}_{Doo}} = 2 \begin{bmatrix} \underline{\Psi}_{Lo} & \underline{\Theta}_{Lo} \\ \underline{\Psi}_{Lo} & -\underline{\Theta}_{Lo} \end{bmatrix}^{-T} \begin{bmatrix} \underline{\mathbf{W}}_{Lo} & \underline{\mathbf{X}}_{Lo} \\ \underline{\mathbf{Y}}_{Lo} & \underline{\mathbf{Z}}_{Lo} \end{bmatrix}^T \begin{bmatrix} \underline{\dot{\mathbf{K}}}_a & \mathbf{0} \\ \mathbf{0} & -\underline{\dot{\mathbf{M}}}_a \end{bmatrix} \begin{bmatrix} \underline{\mathbf{W}}_{Ro} & \underline{\mathbf{X}}_{Ro} \\ \underline{\mathbf{Y}}_{Ro} & \underline{\mathbf{Z}}_{Ro} \end{bmatrix} \begin{bmatrix} \underline{\Psi}_{Ro} & \underline{\Theta}_{Ro} \\ \underline{\Psi}_{Ro} & -\underline{\Theta}_{Ro} \end{bmatrix}^{-1} \quad (7.128)$$

$$\dot{\underline{\mathbf{O}}}_{Koo} = \begin{bmatrix} -\underline{\Theta}_{Lo} & \mathbf{0} \\ (\underline{\Pi}_o \underline{\Psi}_{Lo}) & \underline{\Theta}_{Lo} \end{bmatrix}^{-T} \begin{bmatrix} \underline{\mathbf{W}}_{Lo} & \underline{\mathbf{X}}_{Lo} \\ \underline{\mathbf{Y}}_{Lo} & \underline{\mathbf{Z}}_{Lo} \end{bmatrix}^T \begin{bmatrix} -\dot{\underline{\mathbf{D}}}_a & -\dot{\underline{\mathbf{M}}}_a \\ -\dot{\underline{\mathbf{M}}}_a & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{W}}_{Ro} & \underline{\mathbf{X}}_{Ro} \\ \underline{\mathbf{Y}}_{Ro} & \underline{\mathbf{Z}}_{Ro} \end{bmatrix} \begin{bmatrix} -\underline{\Theta}_{Ro} & \mathbf{0} \\ (\underline{\Pi}_o \underline{\Psi}_{Ro}) & \underline{\Theta}_{Ro} \end{bmatrix}^{-1} \quad (7.129)$$

7.5.5 Residual measures based on mutual orthogonality for second order system

The residuals $(\underline{\mathbf{O}}_{Moo} - \underline{\mathbf{J}})$, $(\underline{\mathbf{O}}_{Doo} - \underline{\mathbf{J}})$ and $(\underline{\mathbf{O}}_{Koo} - \underline{\mathbf{J}})$ are measures of the differences between the systems $\{\underline{\mathbf{K}}_o, \underline{\mathbf{D}}_o, \underline{\mathbf{M}}_o\}$ and $\{\underline{\mathbf{K}}_a, \underline{\mathbf{D}}_a, \underline{\mathbf{M}}_a\}$. If these measures are zeros, then the matrices $\{\underline{\mathbf{M}}_o, \underline{\mathbf{M}}_a\}$, $\{\underline{\mathbf{K}}_o, \underline{\mathbf{K}}_a\}$ and $\{\underline{\mathbf{D}}_o, \underline{\mathbf{D}}_a\}$ are identical pairs, and $\underline{\mathbf{O}}_{Moo} = \underline{\mathbf{J}}$, $\underline{\mathbf{O}}_{Doo} = \underline{\mathbf{J}}$ and $\underline{\mathbf{O}}_{Koo} = \underline{\mathbf{J}}$.

In some cases, there are difficulties in determining the diagonalising transformations directly for the system. These cases might happen if one of the system matrices is singular. The spectral transformations in Chapter 6 are used to find the diagonalising SPEs and the diagonalised system matrices for ill-conditioned systems.

It is obvious that, the three measures in equations (7.105), (7.109) and (7.113) require the invertibility of $\{\underline{\mathbf{M}}_D, \underline{\mathbf{D}}_D, \underline{\mathbf{K}}_D\}$ matrices. In the case of ill-conditioned system matrices, it might be difficult to invert $\{\underline{\mathbf{M}}_D, \underline{\mathbf{D}}_D, \underline{\mathbf{K}}_D\}$. The general ideas for residual measures of undamped systems can be extended to the general damped systems by introducing a measure of the residuals which does not demand invertibility of $\{\underline{\mathbf{M}}_D, \underline{\mathbf{D}}_D, \underline{\mathbf{K}}_D\}$.

7.5.5.1 A palindromic polynomial for second order systems

The developed ideas in section 7.3.5 for undamped systems are extended for general damped second order systems. Matrices which are suggested to match that ideas can be written as

$$\underline{\mathbf{K}}_{LS} = \begin{bmatrix} \underline{\Theta}_{Lo} & \mathbf{0} \\ (\underline{\Pi}_o \underline{\Psi}_{Lo}) & \underline{\Theta}_{Lo} \end{bmatrix}, \quad \underline{\mathbf{K}}_{RS}^T = \begin{bmatrix} \underline{\Theta}_{Ro} & \mathbf{0} \\ (\underline{\Pi}_o \underline{\Psi}_{Ro}) & \underline{\Theta}_{Ro} \end{bmatrix}^T \quad (7.130)$$

$$\underline{\mathbf{D}}_{LS} = \frac{1}{\sqrt{2}} \begin{bmatrix} \underline{\Theta}_{Lo} & \underline{\Psi}_{Lo} \\ -\underline{\Theta}_{Lo} & \underline{\Psi}_{Lo} \end{bmatrix}, \quad \underline{\mathbf{D}}_{RS}^T = \frac{1}{\sqrt{2}} \begin{bmatrix} \underline{\Theta}_{Ro} & -\underline{\Psi}_{Ro} \\ \underline{\Theta}_{Ro} & \underline{\Psi}_{Ro} \end{bmatrix}^T \quad (7.131)$$

$$\underline{\mathbf{M}}_{LS} = \begin{bmatrix} \underline{\Psi}_{Lo} & (\underline{\Pi}_o \underline{\Theta}_{Lo}) \\ \mathbf{0} & \underline{\Psi}_{Lo} \end{bmatrix}, \quad \underline{\mathbf{M}}_{RS}^T = \begin{bmatrix} \underline{\Psi}_{Ro} & (\underline{\Pi}_o \underline{\Theta}_{Ro}) \\ \mathbf{0} & \underline{\Psi}_{Ro} \end{bmatrix}^T \quad (7.132)$$

Pre- and post-multiplying equation (7.79) by $\underline{\mathbf{K}}_{LS}$ and $\underline{\mathbf{K}}_{RS}^T$ in equation (7.130), respectively, pre- and post-multiplying equation (7.80) by $\underline{\mathbf{D}}_{LS}$ and $\underline{\mathbf{D}}_{RS}^T$ in equation (7.131), respectively, and pre- and post-multiplying equation (7.81) by $\underline{\mathbf{M}}_{LS}$ and $\underline{\mathbf{M}}_{RS}^T$ in equation (7.132), respectively, yields

$$\begin{bmatrix} \underline{\Theta}_{Lo} & \mathbf{0} \\ (\underline{\Pi}_o \underline{\Psi}_{Lo}) & \underline{\Theta}_{Lo} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \underline{\mathbf{K}}_{Do} \\ \underline{\mathbf{K}}_{Do} & \underline{\mathbf{D}}_{Do} \end{bmatrix} \begin{bmatrix} \underline{\Theta}_{Ro} & \mathbf{0} \\ (\underline{\Pi}_o \underline{\Psi}_{Ro}) & \underline{\Theta}_{Ro} \end{bmatrix}^T = \begin{bmatrix} \mathbf{0} & \underline{\mathbf{A}} \\ \underline{\mathbf{A}} & \underline{\mathbf{B}} \end{bmatrix} \quad (7.133)$$

$$\frac{1}{2} \begin{bmatrix} \underline{\Theta}_{Lo} & \underline{\Psi}_{Lo} \\ -\underline{\Theta}_{Lo} & \underline{\Psi}_{Lo} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{K}}_{Do} & \mathbf{0} \\ \mathbf{0} & -\underline{\mathbf{M}}_{Do} \end{bmatrix} \begin{bmatrix} \underline{\Theta}_{Ro} & -\underline{\Psi}_{Ro} \\ \underline{\Theta}_{Ro} & \underline{\Psi}_{Ro} \end{bmatrix}^T = \begin{bmatrix} \underline{\mathbf{A}} & \mathbf{0} \\ \mathbf{0} & -\underline{\mathbf{A}} \end{bmatrix} \quad (7.134)$$

$$\begin{bmatrix} \underline{\Psi}_{Lo} & (\underline{\Pi}_o \underline{\Theta}_{Lo}) \\ \mathbf{0} & \underline{\Psi}_{Lo} \end{bmatrix} \begin{bmatrix} -\underline{\mathbf{D}}_{Do} & -\underline{\mathbf{M}}_{Do} \\ -\underline{\mathbf{M}}_{Do} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{\Psi}_{Ro} & (\underline{\Pi}_o \underline{\Theta}_{Ro}) \\ \mathbf{0} & \underline{\Psi}_{Ro} \end{bmatrix}^T = \begin{bmatrix} -\underline{\mathbf{B}} & -\underline{\mathbf{A}} \\ -\underline{\mathbf{A}} & \mathbf{0} \end{bmatrix} \quad (7.135)$$

where

$$\underline{\mathbf{A}} = \underline{\Psi}_{Lo} \underline{\Psi}_{Ro} \underline{\Theta}_{Lo} \underline{\Theta}_{Ro} = \underline{\mathbf{K}}_{Do} \underline{\mathbf{M}}_{Do} \quad (7.136)$$

$$\underline{\mathbf{B}} = \underline{\Pi}_o (\underline{\Psi}_{Lo} \underline{\Theta}_{Ro} + \underline{\Psi}_{Ro} \underline{\Theta}_{Lo}) (\underline{\Psi}_{Lo} \underline{\Psi}_{Ro} + \underline{\Theta}_{Lo} \underline{\Theta}_{Ro}) = \underline{\mathbf{D}}_{Do} (\underline{\mathbf{K}}_{Do} + \underline{\mathbf{M}}_{Do}) \quad (7.137)$$

The matrices $\{\mathbf{A}, \mathbf{B}\}$ represent palindromic polynomials for the second system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$.

7.5.5.2 Residual measures using palindromic polynomial

To avoid the invertibility of $\{\mathbf{M}_D, \mathbf{D}_D, \mathbf{K}_D\}$ while calculating the residuals, equations (7.105), (7.109) and (7.113) are multiplied by matrices. These matrices represent the first and third terms of the left side of equations (7.79), (7.80), (7.81) and each term is multiplied by $\mathbf{K}_{LS}, \mathbf{K}_{RS}^T, \mathbf{D}_{LS}, \mathbf{D}_{RS}^T, \mathbf{M}_{LS}$ and \mathbf{M}_{RS}^T , respectively. These terms can be written as

$$\mathbf{M}_{LI}^T = \begin{bmatrix} \mathbf{\Psi}_{Lo} & (\mathbf{\Pi}_o \mathbf{\Theta}_{Lo}) \\ \mathbf{0} & \mathbf{\Psi}_{Lo} \end{bmatrix}^T, \quad \mathbf{M}_{RI} = \begin{bmatrix} \mathbf{\Psi}_{Ro} & (\mathbf{\Pi}_o \mathbf{\Theta}_{Ro}) \\ \mathbf{0} & \mathbf{\Psi}_{Ro} \end{bmatrix} \quad (7.138)$$

$$\mathbf{D}_{LI}^T = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{\Psi}_{Lo} & \mathbf{\Theta}_{Lo} \\ \mathbf{\Psi}_{Lo} & -\mathbf{\Theta}_{Lo} \end{bmatrix}^T, \quad \mathbf{D}_{RI} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{\Psi}_{Ro} & \mathbf{\Theta}_{Ro} \\ \mathbf{\Psi}_{Ro} & -\mathbf{\Theta}_{Ro} \end{bmatrix} \quad (7.139)$$

$$\mathbf{K}_{LI}^T = \begin{bmatrix} -\mathbf{\Theta}_{Lo} & \mathbf{0} \\ (\mathbf{\Pi}_o \mathbf{\Psi}_{Lo}) & \mathbf{\Theta}_{Lo} \end{bmatrix}^T, \quad \mathbf{K}_{RI} = \begin{bmatrix} -\mathbf{\Theta}_{Ro} & \mathbf{0} \\ (\mathbf{\Pi}_o \mathbf{\Psi}_{Ro}) & \mathbf{\Theta}_{Ro} \end{bmatrix} \quad (7.140)$$

Then, the residuals are calculated by pre/post-multiplying equation (7.105) by $\mathbf{K}_{LS} \mathbf{M}_{LI}^T$ and $\mathbf{M}_{RI} \mathbf{K}_{RS}^T$ in equations (7.130), (7.138), pre/post-multiplying equation (7.109) by $\mathbf{D}_{LS} \mathbf{D}_{LI}^T$ and $\mathbf{D}_{RI} \mathbf{D}_{RS}^T$ in equations (7.131), (7.139), and pre/post-multiplying equation (7.113) by $\mathbf{M}_{LS} \mathbf{K}_{LI}^T$ and $\mathbf{K}_{RI} \mathbf{M}_{RS}^T$ in equations (7.132), (7.140)

$$\mathbf{R}_{Moo} = \left(\mathbf{O}_M - \begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A} & \mathbf{B} \end{bmatrix} \right) \quad (7.141)$$

$$\mathbf{R}_{Doo} = \left(\mathbf{O}_D - \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & -\mathbf{A} \end{bmatrix} \right) \quad (7.142)$$

$$\underline{\mathbf{R}}_{Koo} = \left(\underline{\mathbf{O}}_K - \begin{bmatrix} -\mathbf{B} & -\mathbf{A} \\ -\mathbf{A} & \mathbf{0} \end{bmatrix} \right) \quad (7.143)$$

where

$$\begin{aligned} \underline{\mathbf{O}}_M &= \begin{bmatrix} \mathbf{\Theta}_{Lo} & \mathbf{0} \\ (\mathbf{\Pi}_o \mathbf{\Psi}_{Lo}) & \mathbf{\Theta}_{Lo} \end{bmatrix} \left(\begin{bmatrix} \mathbf{\Psi}_{Lo} & (\mathbf{\Pi}_o \mathbf{\Theta}_{Lo}) \\ \mathbf{0} & \mathbf{\Psi}_{Lo} \end{bmatrix}^T \underline{\mathbf{O}}_{Moo} \begin{bmatrix} \mathbf{\Psi}_{Ro} & (\mathbf{\Pi}_o \mathbf{\Theta}_{Ro}) \\ \mathbf{0} & \mathbf{\Psi}_{Ro} \end{bmatrix} \right) \begin{bmatrix} \mathbf{\Theta}_{Ro} & \mathbf{0} \\ (\mathbf{\Pi}_o \mathbf{\Psi}_{Ro}) & \mathbf{\Theta}_{Ro} \end{bmatrix}^T \\ &= \begin{bmatrix} \mathbf{\Theta}_{Lo} & \mathbf{0} \\ (\mathbf{\Pi}_o \mathbf{\Psi}_{Lo}) & \mathbf{\Theta}_{Lo} \end{bmatrix} \left(\begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K}_a \\ \mathbf{K}_a & \mathbf{D}_a \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} \right) \begin{bmatrix} \mathbf{\Theta}_{Ro} & \mathbf{0} \\ (\mathbf{\Pi}_o \mathbf{\Psi}_{Ro}) & \mathbf{\Theta}_{Ro} \end{bmatrix}^T \end{aligned} \quad (7.144)$$

$$\begin{aligned} \underline{\mathbf{O}}_D &= \frac{1}{4} \begin{bmatrix} \mathbf{\Theta}_{Lo} & \mathbf{\Psi}_{Lo} \\ -\mathbf{\Theta}_{Lo} & \mathbf{\Psi}_{Lo} \end{bmatrix} \left(\begin{bmatrix} \mathbf{\Psi}_{Lo} & \mathbf{\Theta}_{Lo} \\ \mathbf{\Psi}_{Lo} & -\mathbf{\Theta}_{Lo} \end{bmatrix}^T \underline{\mathbf{O}}_{Doo} \begin{bmatrix} \mathbf{\Psi}_{Ro} & \mathbf{\Theta}_{Ro} \\ \mathbf{\Psi}_{Ro} & -\mathbf{\Theta}_{Ro} \end{bmatrix} \right) \begin{bmatrix} \mathbf{\Theta}_{Ro} & -\mathbf{\Psi}_{Ro} \\ \mathbf{\Theta}_{Ro} & \mathbf{\Psi}_{Ro} \end{bmatrix}^T \\ &= \frac{1}{2} \begin{bmatrix} \mathbf{\Theta}_{Lo} & \mathbf{\Psi}_{Lo} \\ -\mathbf{\Theta}_{Lo} & \mathbf{\Psi}_{Lo} \end{bmatrix} \left(\begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_a & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_a \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} \right) \begin{bmatrix} \mathbf{\Theta}_{Ro} & -\mathbf{\Psi}_{Ro} \\ \mathbf{\Theta}_{Ro} & \mathbf{\Psi}_{Ro} \end{bmatrix}^T \end{aligned} \quad (7.145)$$

$$\begin{aligned} \underline{\mathbf{O}}_K &= \begin{bmatrix} \mathbf{\Psi}_{Lo} & (\mathbf{\Pi}_o \mathbf{\Theta}_{Lo}) \\ \mathbf{0} & \mathbf{\Psi}_{Lo} \end{bmatrix} \left(\begin{bmatrix} -\mathbf{\Theta}_{Lo} & \mathbf{0} \\ (\mathbf{\Pi}_o \mathbf{\Psi}_{Lo}) & \mathbf{\Theta}_{Lo} \end{bmatrix}^T \underline{\mathbf{O}}_{Koo} \begin{bmatrix} -\mathbf{\Theta}_{Ro} & \mathbf{0} \\ (\mathbf{\Pi}_o \mathbf{\Psi}_{Ro}) & \mathbf{\Theta}_{Ro} \end{bmatrix} \right) \begin{bmatrix} \mathbf{\Psi}_{Ro} & (\mathbf{\Pi}_o \mathbf{\Theta}_{Ro}) \\ \mathbf{0} & \mathbf{\Psi}_{Ro} \end{bmatrix}^T \\ &= \begin{bmatrix} \mathbf{\Psi}_{Lo} & (\mathbf{\Pi}_o \mathbf{\Theta}_{Lo}) \\ \mathbf{0} & \mathbf{\Psi}_{Lo} \end{bmatrix} \left(\begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix}^T \begin{bmatrix} -\mathbf{D}_a & -\mathbf{M}_a \\ -\mathbf{M}_a & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} \right) \begin{bmatrix} \mathbf{\Psi}_{Ro} & (\mathbf{\Pi}_o \mathbf{\Theta}_{Ro}) \\ \mathbf{0} & \mathbf{\Psi}_{Ro} \end{bmatrix}^T \end{aligned} \quad (7.146)$$

where the matrices $\underline{\mathbf{O}}_{Moo}$, $\underline{\mathbf{O}}_{Doo}$ and $\underline{\mathbf{O}}_{Koo}$ are the same as equations (7.105), (7.109)

and (7.113), respectively.

It is appropriate to vectorise $\underline{\mathbf{R}}_{Moo}$, $\underline{\mathbf{R}}_{Doo}$ and $\underline{\mathbf{R}}_{Koo}$ into the form $\mathbf{r}_M, \mathbf{r}_D, \mathbf{r}_K$. The total residual can be written in the form

$$\mathbf{r} = \begin{bmatrix} \mathbf{r}_M \\ \mathbf{r}_D \\ \mathbf{r}_K \end{bmatrix} \quad (7.147)$$

The rate of change of equations (7.144), (7.145) and (7.146) can be written as

$$\dot{\underline{\mathbf{O}}}_M = \begin{bmatrix} \mathbf{\Theta}_{Lo} & \mathbf{0} \\ (\mathbf{\Pi}_o \mathbf{\Psi}_{Lo}) & \mathbf{\Theta}_{Lo} \end{bmatrix} \left(\begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \dot{\mathbf{K}}_a \\ \dot{\mathbf{K}}_a & \dot{\mathbf{D}}_a \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} \right) \begin{bmatrix} \mathbf{\Theta}_{Ro} & \mathbf{0} \\ (\mathbf{\Pi}_o \mathbf{\Psi}_{Ro}) & \mathbf{\Theta}_{Ro} \end{bmatrix}^T \quad (7.148)$$

$$\dot{\underline{\mathbf{O}}}_D = \frac{1}{2} \begin{bmatrix} \mathbf{\Theta}_{Lo} & \mathbf{\Psi}_{Lo} \\ -\mathbf{\Theta}_{Lo} & \mathbf{\Psi}_{Lo} \end{bmatrix} \left(\begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix}^T \begin{bmatrix} \dot{\mathbf{K}}_a & \mathbf{0} \\ \mathbf{0} & -\dot{\mathbf{M}}_a \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} \right) \begin{bmatrix} \mathbf{\Theta}_{Ro} & -\mathbf{\Psi}_{Ro} \\ \mathbf{\Theta}_{Ro} & \mathbf{\Psi}_{Ro} \end{bmatrix}^T \quad (7.149)$$

$$\dot{\underline{\mathbf{O}}}_K = \begin{bmatrix} \underline{\Psi}_{Lo} & (\underline{\Pi}_o \underline{\Theta}_{Lo}) \\ \mathbf{0} & \underline{\Psi}_{Lo} \end{bmatrix} \left(\begin{bmatrix} \underline{\mathbf{W}}_{Lo} & \underline{\mathbf{X}}_{Lo} \\ \underline{\mathbf{Y}}_{Lo} & \underline{\mathbf{Z}}_{Lo} \end{bmatrix}^T \begin{bmatrix} -\dot{\underline{\mathbf{D}}}_a & -\dot{\underline{\mathbf{M}}}_a \\ -\dot{\underline{\mathbf{M}}}_a & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{W}}_{Ro} & \underline{\mathbf{X}}_{Ro} \\ \underline{\mathbf{Y}}_{Ro} & \underline{\mathbf{Z}}_{Ro} \end{bmatrix} \right) \begin{bmatrix} \underline{\Psi}_{Ro} & (\underline{\Pi}_o \underline{\Theta}_{Ro}) \\ \mathbf{0} & \underline{\Psi}_{Ro} \end{bmatrix}^T \quad (7.150)$$

These derivatives are independent of any information from system $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$.

7.6 Example

We consider a system having four degrees of freedom with general linear damping.

The system matrices $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ have been deliberately chosen to have a non symmetric and singular mass matrix. The information required are the system matrices for the analytical system $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$. For the measured system, the diagonalised system matrices $\{\mathbf{K}_{Do}, \mathbf{D}_{Do}, \mathbf{M}_{Do}\}$ and the diagonalising transformations

$\{\underline{\mathbf{T}}_{Lo}, \underline{\mathbf{T}}_{Ro}\}$ can be calculated by using spectral transformations. The measured system matrices $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ are

$$\mathbf{K}_o = \begin{bmatrix} 4342 & -2621 & 3704 & -698 \\ -2356 & 1458 & -4901 & 3903 \\ -3397 & 4669 & -3630 & 2349 \\ 3729 & 1649 & 3188 & 1873 \end{bmatrix}$$

$$\mathbf{D}_o = \begin{bmatrix} -15 & -8 & -4 & 40 \\ -33 & 36 & -4 & -49 \\ -34 & -1 & -5 & -20 \\ -31 & 32 & -9 & -45 \end{bmatrix} \quad \mathbf{M}_o = \begin{bmatrix} 193 & -100 & -124 & 0 \\ 150 & -301 & -490 & 0 \\ 483 & 125 & -80 & 0 \\ 53 & 233 & 254 & 0 \end{bmatrix}$$

The eigenvalues are

$$\Lambda = \text{diag} \left(\begin{bmatrix} 0.0430 \pm 0.7679i \\ 0.0080 \pm 2.6493i \\ 0.4233 \pm 13.3734i \\ 54.95495 \\ Inf \end{bmatrix} \right)$$

The diagonalising SPE's $\{\underline{\mathbf{T}}_{Ro}, \underline{\mathbf{T}}_{Lo}\}$ for the measured system are calculated using

Möbius transformation. The diagonalised system matrices $\{\mathbf{K}_{Do}, \mathbf{D}_{Do}, \mathbf{M}_{Do}\}$ are

$$\mathbf{K}_{Do} = \text{diag} \begin{bmatrix} -0.6238 \\ -0.60378 \\ -0.2872 \\ -0.6197 \end{bmatrix}, \quad \mathbf{D}_{Do} = \text{diag} \begin{bmatrix} 0.00295 \\ 0.00138 \\ 0.04179 \\ 0.01128 \end{bmatrix}, \quad \mathbf{M}_{Do} = \text{diag} \begin{bmatrix} -0.00348 \\ -0.08601 \\ -0.48541 \\ 0.00000 \end{bmatrix}$$

The i^{th} entry of the matrices \mathbf{M}_{Do} , \mathbf{K}_{Do} and \mathbf{D}_{Do} are written as m_i , k_i and d_i . The right and left matrices $\{\Theta_R, \Theta_L\}$, $\{\Psi_R, \Psi_L\}$ can be calculated. If the i^{th} entry of $\text{sign}(m_i k_i) < 0$, is less than zero, the i^{th} entry of the right and left matrices $\{\Theta_R, \Theta_L\}$, $\{\Psi_R, \Psi_L\}$ are chosen as

$$\begin{aligned} \Theta_{Ri} &= 1, & \Theta_{Li} &= m_i / \Theta_{Ri} \\ \Psi_{Ri} &= \sqrt{\text{abs}(k_i)}, & \Psi_{Li} &= k_i / \Psi_{Ri} \end{aligned}$$

If the i^{th} entry of $\text{sign}(m_i k_i) > 0$, is greater than zero, the i^{th} entry of the right and left matrices $\{\Theta_R, \Psi_R\}$, $\{\Theta_L, \Psi_L\}$ are equal

$$\begin{aligned} \Theta_{Ri} &= \sqrt{\text{abs}(m_i)}, & \Theta_{Li} &= m_i / \Theta_{Ri} \\ \Psi_{Ri} &= \sqrt{\text{abs}(k_i)}, & \Psi_{Li} &= k_i / \Psi_{Ri} \end{aligned}$$

Moreover,

$$\begin{aligned} \Theta_{Ri} &= 1, & \Theta_{Li} &= m_i / \Theta_{Ri} \\ \Psi_{Ri} &= 1, & \Psi_{Li} &= k_i / \Psi_{Ri} \end{aligned}$$

The i^{th} entry of matrix $\mathbf{\Pi}$ can be calculated as

$$\Pi_i = (\Psi_{Li} \Theta_{Ri} + \Psi_{Ri} \Theta_{Li}) / d_i$$

The analytical system matrices $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$ are

$$\mathbf{K}_a = \begin{bmatrix} 3937.5148 & -2424.9165 & 3495.5983 & -894.0835 \\ -2159.9165 & 1261.9165 & -4901 & 4099.0835 \\ -3605.4017 & 4669 & -3838.4017 & 2349 \\ 3532.9165 & 1845.0835 & 3188 & 1676.9165 \end{bmatrix}$$

$$\mathbf{D}_a = \begin{bmatrix} 43.8749 & 530.9069 & 593.7819 & -498.9069 \\ 505.9069 & -502.9069 & -4 & 489.9069 \\ 563.7819 & -1 & 592.7819 & -20 \\ -569.9069 & 570.9069 & -9 & -583.9069 \end{bmatrix}$$

$$\mathbf{M}_a = \begin{bmatrix} 259.0495 & -499.2043 & -457.1547 & 399.2043 \\ -249.2042 & 98.2043 & -490 & -399.2043 \\ 149.8453 & 125 & -413.1547 & 0 \\ 452.2043 & -166.2043 & 254 & 399.2043 \end{bmatrix}$$

The eigenvalues of the analytical system are found

$$\Lambda_a = \begin{bmatrix} 0.02154 \pm 0.7922i \\ -2.62184 \\ 2.20235 \pm 2.9523i \\ -2.07063 \pm 3.05288i \\ 5.9091 \end{bmatrix}$$

The aim in this example is to match the measured modal information with that from the analytical system. The matching used is based on *mutual orthogonality* measures as in equations (7.105), (7.109) and (7.113) to satisfy bi-**J**-unitary through the system matrices $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$ by driving the residuals to zero.

This example does not require invertibility of matrices $\{\mathbf{M}_D, \mathbf{D}_D, \mathbf{K}_D\}$. Equations (7.141), (7.142) and (7.143) are used to calculate the residuals. The matrices $\{\mathbf{A}, \mathbf{B}\}$ are calculated as in equations (7.136) and (7.137)

$$\mathbf{A} = \text{diag} \left(\begin{bmatrix} 0.00217 \\ 0.05193 \\ 0.13938 \\ 0.00000 \end{bmatrix} \right), \quad \mathbf{B} = \text{diag} \left(\begin{bmatrix} -0.00185 \\ -0.00095 \\ -0.03228 \\ -0.00699 \end{bmatrix} \right)$$

The sensitivity matrix is calculated at each change of the scalar parameters $\{a, b, c, d, e, f\}$ in the vector $\mathbf{\theta}$.

7.7 Conclusion

This chapter has developed modal correlation methods for undamped systems. The general ideas of modal correlation for undamped systems have been extended for generally damped systems. *Cross orthogonality* measures are performed as modal correlation methods. The product of different cross orthogonality measures has been referred as *mutual orthogonality* measures. Both *cross orthogonality* and *mutual orthogonality* measures for generally damped linear systems have been presented in terms of *diagonalising structure preserving equivalences* (DSPE's). In this chapter, a new measure of residuals has been investigated which does not demand invertibility of the matrices $\{\underline{\mathbf{M}}_D, \underline{\mathbf{D}}_D, \underline{\mathbf{K}}_D\}$.

Cross orthogonality measures require matching the right and left diagonalising SPEs from the computed system with that from the true system. In contrast, the *mutual orthogonality* measures do not involve this matching. However, both left and right modes are required for the *mutual orthogonality measures*. In case of ill-conditioned systems such that the mass matrix is singular, the diagonalising SPEs can be calculated using new methods in chapter 6 and the methods in appendices C and D.

The following chapter addresses model updating based on the methods developed in this chapter.

CHAPTER 8. Model Updating

This chapter presents model updating methods for generally damped systems based on modal correlation methods in the previous chapter. Model updating techniques are commonly used to adjust selected parameters of finite element (FE) models in order to match models with experimental data. Most experience in model updating is based on natural frequencies and mode shapes. The natural frequencies and mode shapes are identified from vibration test data. Experimental modal analysis results are often used to evaluate the predicted model information from FE analysis predictions.

If the number of parameters becomes too large, the effects of parameter changes on the FE analysis results are not easy to assess and lengthy computational model updating procedures must be used. These procedures are able to adjust several parameters such that the difference between measured and predicted natural frequencies and mode shapes are minimised [90]. Numerous different methods are used to update model parameters.

In this chapter we focus on developing model updating methods for undamped systems, extend the ideas for generally damped systems and compare the developed modal correlation methods with alternative methods for model updating. Three related measures of error are presented which make it possible to determine what adjustments should be applied to the system mass, damping and stiffness matrices.

8.1 Model updating

The differences between analytical data and experimental data allow for an evaluation of the quality of the utilized finite element model. For general damped systems, we consider that the system matrices $\{\mathbf{K}_a(\boldsymbol{\theta}), \mathbf{D}_a(\boldsymbol{\theta}), \mathbf{M}_a(\boldsymbol{\theta})\}$ are functions of a parameter vector $\boldsymbol{\theta}$ containing p entries. The original system matrices are $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$. In effect, we are assuming that stiffness, damping and mass matrices can change continuously as the following equations indicate

$$\mathbf{K}_a(\boldsymbol{\theta}) = \mathbf{K}_o + \theta_1 \mathbf{K}_1 + \theta_2 \mathbf{K}_2 + \dots + \theta_p \mathbf{K}_p = \mathbf{K}_o + \sum_{j=1}^p \theta_j \mathbf{K}_j \quad (8.1)$$

$$\mathbf{D}_a(\boldsymbol{\theta}) = \mathbf{D}_o + \theta_1 \mathbf{D}_1 + \theta_2 \mathbf{D}_2 + \dots + \theta_p \mathbf{D}_p = \mathbf{D}_o + \sum_{j=1}^p \theta_j \mathbf{D}_j \quad (8.2)$$

$$\mathbf{M}_a(\boldsymbol{\theta}) = \mathbf{M}_o + \theta_1 \mathbf{M}_1 + \theta_2 \mathbf{M}_2 + \dots + \theta_p \mathbf{M}_p = \mathbf{M}_o + \sum_{j=1}^p \theta_j \mathbf{M}_j \quad (8.3)$$

The subscripts “ o ” and “ a ” refer to the measured and analytical systems.

Since matrices $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$ depend on a parameter vector $\boldsymbol{\theta}$, the eigenvalues and their corresponding eigenvectors also depend on $\boldsymbol{\theta}$. Both the diagonalising transformation and the diagonalised system change correspondingly (see Chapter 5). Then the derivatives of the diagonalised system and diagonalising SPEs can be calculated. The rates of change of eigenvalues and eigenvectors can be solved from these.

There are numerous methods in the literature to determine eigenvalue and eigenvector derivatives for general non-symmetric undamped systems [97-104].

Computing the derivative of any one eigenvector of an undamped system requires only the eigenvalue and eigenvector under consideration [104]. The methods of complex eigenvalue and eigenvector derivatives are derived and developed for general asymmetric second-order damped systems [125-128, 136]. The derivatives of the diagonalised system matrices and diagonalising SPEs for the damped system in equations (8.1),(8.2) and (8.3) are presented [121, 135] for a single scalar parameter θ_j .

8.1.1 Residuals and the sensitivity matrix

Model updating methods are based on the definition of a vector of parameters and a vector of residuals [137, 138]. Also, model updating methods use a sensitivity matrix relating changes in model outputs to changes in model parameters in order to determine what changes in the parameters will reduce the overall discrepancy. The finite element model is then reformed using the new values of the updated parameters, and the process repeats until some convergence criteria are met [139]. The model parameters are adjusted iteratively until the analytical output data matches the measured data satisfactorily. Many different methods have been proposed for updating the model parameters. All of these attempted to minimise differences between computed and measured data. These differences are called residuals. The residuals can be written as

$$\mathbf{R} = (\mathbf{O}_o - \mathbf{O}_a) \quad (8.4)$$

Matrices $\{\mathbf{O}_o, \mathbf{O}_a\}$ are the physical and analytical output data respectively. It is appropriate to vectorise \mathbf{R} to define an output vector, $\Delta \mathbf{r}$

$$\mathbf{r} = \text{vec}(\mathbf{R}) \quad (8.5)$$

In general, the vector \mathbf{r} may include some complex values. In this work we shall assume without loss of generality that \mathbf{r} contains only real values. There are two reasons for this. Firstly where any complex values arise, we can replace the single complex number $(a + jb)$ by the two real numbers $\{a, b\}$. Secondly, the new model updating methods proposed here never utilise any complex quantity.

Now, since \mathbf{R} depends on $\boldsymbol{\theta}$, then the vector of residuals \mathbf{r} is also a function of vector parameters $\boldsymbol{\theta}$. We can define the weighted sum of squares of residuals E , according to

$$E = (\mathbf{r}^T \mathbf{W} \mathbf{r}) \quad (8.6)$$

where \mathbf{W} is a positive definite weighting matrix.

For a given vector of parameters $\boldsymbol{\theta}_0$ we can calculate the corresponding $\Delta \mathbf{r}(\boldsymbol{\theta}_0) =: \Delta \mathbf{r}_0$. The solution to the problem of minimising E usually leads to set of nonlinear equations because $\mathbf{r}(\boldsymbol{\theta})$ is in general nonlinearly dependent on the estimation parameters. The objective function E can be minimized iteratively. The truncated Taylor series for $\mathbf{r}(\boldsymbol{\theta})$ is given by

$$\mathbf{r}(\boldsymbol{\theta}) := \mathbf{r}_0 + \Delta \mathbf{r} \quad (8.7)$$

and for some real sensitivity matrix, \mathbf{S} ,

$$\Delta \mathbf{r} = \mathbf{S} \Delta \boldsymbol{\theta} \quad (8.8)$$

Since we have a numerical model, it is straightforward to determine the matrix \mathbf{S} numerically. For each entry of vector parameters $\boldsymbol{\theta}$, a small positive change δ is made in the entry and the corresponding \mathbf{r}^+ is computed. Then a small negative

change $-\delta$ is made in the same entry and the corresponding \mathbf{r}^- is computed. Finally, each column of the sensitivity matrix \mathbf{S} can be calculated numerically using the central difference formula

$$\mathbf{s}_c = \frac{(\mathbf{r}^+ - \mathbf{r}^-)}{2\delta} \quad (8.9)$$

The sensitivity matrix in equation (8.8) can be computed analytically by calculating the derivatives of the analytical output data \mathbf{O}_a with respect to each single scalar parameter in vector $\boldsymbol{\theta}$.

$$\Delta \mathbf{O}_a = \sum_{j=1}^p \frac{\partial \mathbf{O}_a}{\partial \theta_j} \Delta \theta_j \quad (8.10)$$

Equation (8.8) may be fully determined, over-determined or under-determined depending on whether the number of output data is equal to, larger than or smaller than the number of scalar parameters $\boldsymbol{\theta}$. In any case, the change of parameter $\Delta \boldsymbol{\theta}$ in equation (8.8) can be solved using the pseudo inverse [140]. To obtain

$$\Delta \boldsymbol{\theta} = -\mathbf{S}^+ \Delta \mathbf{y} \quad (8.11)$$

where \mathbf{S}^+ is the left pseudo-inverse matrix of the sensitivity matrix \mathbf{S}

8.1.1.1 *Solution of under-determined or ill-conditioned equations*

In cases where the equation (8.8) is under-determined or ill-conditioned, there are two possible approaches: singular value decomposition (SVD) and regularisation. The benefits of these methods in model updating are that in both cases their application results in the unique solution of $\Delta \boldsymbol{\theta}$ [18].

Singular value decomposition SVD is used in under-determined problems and it results in the unique minimum norm solution.

$$\mathbf{S} = \mathbf{U} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \mathbf{V}^T \quad (8.12)$$

consider

$$\|\Delta \mathbf{r} - \mathbf{S} \Delta \boldsymbol{\theta}\|^2 = \left\| \mathbf{U}^T (\Delta \mathbf{r} - \mathbf{S} \mathbf{V} \mathbf{V}^T \Delta \boldsymbol{\theta}) \right\|^2 \quad (8.13)$$

The parameter vector $\boldsymbol{\theta}$ can be obtained in terms of the singular values of the right and left singular vectors

$$\Delta \boldsymbol{\theta} = \mathbf{V} \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} \mathbf{U}^T \Delta \mathbf{r} \quad (8.14)$$

Regularisation is an alternative method used in model updating to solve under-determined or ill conditioned equations. This approach involves minimising a cost function

$$J_\lambda(\boldsymbol{\theta}) = \|\mathbf{S} \Delta \boldsymbol{\theta} - \Delta \mathbf{r}\|^2 + \lambda^2 \|\mathbf{B} \Delta \boldsymbol{\theta}\|^2 \quad (8.15)$$

where λ is the regularisation parameter. The matrix \mathbf{B} has full rank. The generalised solution depends on λ . The derivatives of the cost function with respect to $\boldsymbol{\theta}$ can be written as

$$(\mathbf{S}^T \mathbf{S} + \lambda^2 \mathbf{B}^T \mathbf{B}) \Delta \boldsymbol{\theta} = \mathbf{S}^T \Delta \mathbf{r} \quad (8.16)$$

The singular value decomposition approach is a particular case of regularisation where $(\mathbf{B}^T \mathbf{B})$ is the identity matrix and λ^2 is infinity.

8.2 Model updating for undamped system

Most experience in model updating is based on natural frequencies and mode shapes. Eigenvalues and the corresponding eigenvectors are often identified from vibration modal testing. However, the computed modal information is obtained from finite

element methods. The discrepancies between measured and computed modal information are used to evaluate and qualify the predicted model information.

We are assuming that stiffness and mass matrices $\{\mathbf{K}_a, \mathbf{M}_a\}$ are changing continuously according to equations (8.1) and (8.3).

8.2.1 *Model updating based on eigenpairs of undamped system*

8.2.1.1 *Eigenvalues and eigenvectors for undamped systems*

The general governing equation of motion for an undamped second order system is introduced in Chapter 2, section 2.3.1.

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f} \quad (8.17)$$

The eigenvalues $\Lambda = \text{diag}(\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2)$ might be complex or real or mixed (real and complex). The corresponding left and right eigenvector matrices $\{\Phi_L, \Phi_R\}$ diagonalise the system matrices $\{\mathbf{K}, \mathbf{M}\}$. The eigenvectors are commonly mass normalised in which case they satisfy the following equations

$$\Phi_L^T \mathbf{M} \Phi_R = \mathbf{I} \quad (8.18)$$

$$\Phi_L^T \mathbf{K} \Phi_R = \Lambda \quad (8.19)$$

The i^{th} eigenvalue λ_i with the corresponding left and right eigenvectors $\{\Phi_{Li}, \Phi_{Ri}\}$ satisfy the following equations

$$(\mathbf{K} - \lambda_i^2 \mathbf{M})\Phi_{Ri} = 0 \quad , \quad \Phi_{Li}^T (\mathbf{K} - \lambda_i^2 \mathbf{M}) = 0 \quad (8.20)$$

8.2.1.2 Changing of eigenpairs for undamped system

Changes in the system matrices $\{\mathbf{K}_a, \mathbf{M}_a\}$ cause direct effects in the dynamic properties. Obviously, the eigenvalues and their corresponding eigenvectors are also changed.

$$\lambda_i = \lambda_{oi} + \Delta \boldsymbol{\theta}^T \frac{\partial \lambda_i}{\partial \boldsymbol{\theta}} \quad (8.21)$$

$$\boldsymbol{\phi}_{Ri} = \boldsymbol{\phi}_{Roi} + \frac{\partial \boldsymbol{\phi}_{Ri}}{\partial \boldsymbol{\theta}} \Delta \boldsymbol{\theta} \quad , \quad \boldsymbol{\phi}_{Li} = \boldsymbol{\phi}_{Loi} + \frac{\partial \boldsymbol{\phi}_{Li}}{\partial \boldsymbol{\theta}} \Delta \boldsymbol{\theta} \quad (8.22)$$

An estimate of the discrepancies between the physical modal information $\{\boldsymbol{\Phi}_{Ro}, \boldsymbol{\Lambda}_o, \boldsymbol{\Phi}_{Lo}\}$ and computed modal information $\{\boldsymbol{\Phi}_{Ra}, \boldsymbol{\Lambda}_a, \boldsymbol{\Phi}_{La}\}$ can be achieved by computing the sensitivity of this modal information to changes in the parameters. This can be done either numerically or analytically. The rates of change of eigenvalues and eigenvectors for undamped systems are introduced in Chapter 5, section 5.2 and section 5.5.

8.2.1.3 Scaling and ordering Eigenvectors of undamped systems

The measured eigenvectors are compared with the predicted eigenvectors according to some rules. The analytical eigenvectors are scaled and ordered such that the resulting ordered and scaled vectors are like eigenvectors of the measured system.

Mass normalisation is applied to both systems.

$$\begin{aligned} \mathbf{X}_o &= \boldsymbol{\Phi}_{Lo} \mathbf{M}_a \boldsymbol{\Phi}_{Ro} \\ \mathbf{X}_a &= \boldsymbol{\Phi}_{La} \mathbf{M}_a \boldsymbol{\Phi}_{Ra} \end{aligned} \quad (8.23)$$

The right and left eigenvectors for both systems are scaled accordingly. Rewrite equation (8.23) with scaled eigenvectors,

$$\begin{aligned}\mathbf{X}_{so} &= \mathbf{\Phi}_{Lo} \mathbf{M}_a \mathbf{\Phi}_{Ro} \\ \mathbf{X}_{sa} &= \mathbf{\Phi}_{La} \mathbf{M}_a \mathbf{\Phi}_{Ra}\end{aligned}\quad (8.24)$$

Balance the scaling of eigenvectors of the analytical system

$$\mathbf{t}_o = \text{diag}(\mathbf{X}_{so} \mathbf{X}_{so}^T) \quad , \quad \mathbf{t}_a = \text{diag}(\mathbf{X}_{sa} \mathbf{X}_{sa}^T) \quad , \quad \mathbf{t}_b = \text{diag}\left(\sqrt{\sqrt{(\mathbf{t}_o / \mathbf{t}_a)}}\right) \quad (8.25)$$

The balanced analytical eigenvectors can be written as

$$\mathbf{\Phi}_{Rba} = \mathbf{\Phi}_{Rsa} / \mathbf{t}_b \quad , \quad \mathbf{\Phi}_{Lba} = \mathbf{\Phi}_{Lsa} \mathbf{t}_b \quad (8.26)$$

$$\mathbf{X}_{bo} = \mathbf{X}_{so} / \mathbf{t}_b \quad , \quad \mathbf{X}_{ba} = \mathbf{X}_{sa} \mathbf{t}_b \quad (8.27)$$

The sum of the generalised mass matrices $\{\mathbf{X}_{bo}, \mathbf{X}_{ba}\}$ can be written as

$$\mathbf{\mathfrak{X}} = \mathbf{X}_{bo} + \mathbf{X}_{ba} \quad (8.28)$$

Each column in the analytical eigenvector matrices $\{\mathbf{\Phi}_{Rba}, \mathbf{\Phi}_{Lba}\}$ is sorted according to locations of the largest values in the rows of the generalised mass matrix $\mathbf{\mathfrak{X}}$.

$$[\mathfrak{Y} \quad \mathfrak{L}] = \max(\mathbf{\mathfrak{X}}^T) \quad (8.29)$$

$$\mathbf{\Phi}_{Ra} = \mathbf{\Phi}_{Rba}(:, \mathfrak{L}) \quad , \quad \mathbf{\Phi}_{La} = \mathbf{\Phi}_{Lba}(:, \mathfrak{L}) \quad (8.30)$$

The \mathfrak{Y} has dimension $(1 \times n)$ and defines the maximum values in the rows of matrix $\mathbf{\mathfrak{X}}$. The \mathfrak{L} has dimension $(1 \times n)$ and defines the locations of the largest values \mathfrak{Y} in the rows of matrix $\mathbf{\mathfrak{X}}$.

8.2.1.4 Residual measures based on eigenpairs of undamped systems

Model updating methods based on eigenpairs are used to minimise the discrepancies between the measured and predicted output data. The differences between the

measured and computed modal information is used in the vector of residuals \mathbf{r} . The vector of residuals can be written as

$$\mathbf{r} = \begin{bmatrix} \text{vec}(\Phi_{Ro} - \Phi_{Ra}) \\ \text{vec}(\Phi_{Lo} - \Phi_{La}) \\ \text{diag}(\Lambda_o - \Lambda_a) \end{bmatrix} \quad (8.31)$$

The vector \mathbf{r} may include some complex values. However, in this work we assume that \mathbf{r} contains only real values.

8.2.2 *Model updating based on cross orthogonality measures of undamped systems*

Changing the system matrices $\{\mathbf{K}_a, \mathbf{M}_a\}$ also causes a change in the corresponding *cross orthogonality* measures. *Cross orthogonality* measures and their derivatives for undamped systems are introduced in Chapter 7 section 7.3.

$$\mathbf{XO}_{Ma} = \mathbf{XO}_M + \Delta \mathbf{XO}_M = \mathbf{XO}_M + \sum_{j=1}^p \frac{\partial \mathbf{XO}_{Ma}}{\partial \theta_j} \Delta \theta_j \quad (8.32)$$

$$\mathbf{XO}_{Ka} = \mathbf{XO}_K + \Delta \mathbf{XO}_K = \mathbf{XO}_K + \sum_{j=1}^p \frac{\partial \mathbf{XO}_{Ka}}{\partial \theta_j} \Delta \theta_j \quad (8.33)$$

The product of two mass-weighted *cross orthogonality* measures is defined in equation (7.43) as

$$\mathbf{XO}_{Ma} = (\Theta_{Lo}^{-1} \Phi_{Lo}^T \mathbf{M}_a \Phi_{Ra} \Theta_{Ra}^{-1}) \times (\Theta_{La}^{-1} \Phi_{La}^T \mathbf{M}_a \Phi_{Ro} \Theta_{Ro}^{-1}) \quad (8.34)$$

Also the product of two different stiffness-weighted *cross-orthogonality* measures is expressed in equation (7.45) as

$$\mathbf{XO}_{Ka} = (\Psi_{Lo}^{-1} \Phi_{Lo}^T \mathbf{K}_a \Phi_{Ra} \Psi_{Ra}^{-1}) \times (\Psi_{La}^{-1} \Phi_{La}^T \mathbf{K}_a \Phi_{Ro} \Psi_{Ro}^{-1}) \quad (8.35)$$

Equations (8.34) and (8.35) are used for updating model parameters. Full model information is required for both systems. The analytical eigenvectors are scaled and

ordered to match the measured eigenvectors. The same procedures introduced in section 8.2.1.3 are used for scaling and ordering the analytical eigenvectors.

8.2.2.1 *Residual measures based on cross orthogonality measures of undamped systems*

Model updating based on *cross orthogonality* can be used to minimise the differences between the measured model information and analytical model information. The vector of residuals \mathbf{r} has dimension $(2n^2)$. The residuals $(\mathbf{XO}_{Ka} - \mathbf{I}_n)$ and $(\mathbf{XO}_{Ma} - \mathbf{I}_n)$ are measures of the differences between systems $\{\mathbf{K}_o, \mathbf{M}_o\}$ and $\{\mathbf{K}_a, \mathbf{M}_a\}$. The vector of residuals is expressed as

$$\mathbf{r} = \begin{bmatrix} \text{vec}(\mathbf{XO}_{Ka} - \mathbf{I}_n) \\ \text{vec}(\mathbf{XO}_{Ma} - \mathbf{I}_n) \end{bmatrix} \quad (8.36)$$

where \mathbf{I}_n defines the identity matrix.

Computing the sensitivity of *cross orthogonality* measures can be done either numerically or analytically. The derivatives of *cross orthogonality* measures are introduced in Chapter 7, section 7.3.4.1.

In this section all full modal information for both systems are required. The eigenvectors of the analytical system are scaled and ordered correctly to match the eigenvectors of the measured system.

8.2.3 Model updating based on mutual orthogonality measures of undamped systems

Mutual orthogonality measures are introduced in Chapter 7 section 7.3. The product of the *cross orthogonality* measures in equations (8.34) and (8.35) defines the *mutual orthogonality* measures. These measures are independent of any information from the system $\{\mathbf{K}_a, \mathbf{M}_a\}$.

$$\mathbf{O}_{Moo} = (\mathbf{\Theta}_{Lo}^{-1} \mathbf{\Phi}_{Lo}^T) \times \mathbf{M}_a \times (\mathbf{\Phi}_{Ro} \mathbf{\Theta}_{Ro}^{-1}) \quad (8.37)$$

$$\mathbf{O}_{Koo} = (\mathbf{\Psi}_{Lo}^{-1} \mathbf{\Phi}_{Lo}^T) \times \mathbf{K}_a \times (\mathbf{\Phi}_{Ro} \mathbf{\Psi}_{Ro}^{-1}) \quad (8.38)$$

The above two measures in equations (8.37) and (8.38) depend on the invertibility of the diagonalised system matrices $\{\mathbf{M}_{Do}, \mathbf{K}_{Do}\}$. A measure is introduced in Chapter 7, section 7.3.5 which does not demand this invertibility.

$$\mathbf{\Psi}_{Lo} \mathbf{M}_{Do} \mathbf{\Psi}_{Ro} = \mathbf{\Psi}_{Lo} \mathbf{\Theta}_{Lo} \mathbf{\Theta}_{Ro} \mathbf{\Psi}_{Ro} \quad (8.39)$$

$$\mathbf{\Theta}_{Lo} \mathbf{K}_{Do} \mathbf{\Theta}_{Ro} = \mathbf{\Theta}_{Lo} \mathbf{\Psi}_{Lo} \mathbf{\Psi}_{Ro} \mathbf{\Theta}_{Ro} \quad (8.40)$$

The matrices $\{\mathbf{\Psi}_{Lo}, \mathbf{\Psi}_{Ro}, \mathbf{\Theta}_{Lo}, \mathbf{\Theta}_{Ro}\}$ are diagonal. Then, equations (8.39) and (8.40) are identical

$$\mathbf{A} =: \mathbf{\Psi}_{Lo} \mathbf{\Theta}_{Lo} \mathbf{\Theta}_{Ro} \mathbf{\Psi}_{Ro} \quad (8.41)$$

Equations (8.37) and (8.38) are modified and can be written as

$$\mathbf{O}_M = \mathbf{\Psi}_{Lo} (\mathbf{\Theta}_{Lo} \mathbf{O}_{Moo} \mathbf{\Theta}_{Ro}) \mathbf{\Psi}_{Ro} = \mathbf{\Psi}_{Lo} (\mathbf{\Phi}_{Lo}^T \mathbf{M}_a \mathbf{\Phi}_{Ro}) \mathbf{\Psi}_{Ro} \quad (8.42)$$

$$\mathbf{O}_K = \mathbf{\Theta}_{Lo} (\mathbf{\Psi}_{Lo} \mathbf{O}_{Koo} \mathbf{\Psi}_{Ro}) \mathbf{\Theta}_{Ro} = \mathbf{\Theta}_{Lo} (\mathbf{\Phi}_{Lo}^T \mathbf{K}_a \mathbf{\Phi}_{Ro}) \mathbf{\Theta}_{Ro} \quad (8.43)$$

8.2.3.1 Residual measures based on mutual orthogonality measures of undamped systems

Model updating based on *mutual orthogonality* measures is used to minimise the discrepancies between the measured and analytical model information. The residuals $(\mathbf{O}_M - \mathbf{A})$ and $(\mathbf{O}_K - \mathbf{A})$ are measures of the differences between the systems $\{\mathbf{K}_o, \mathbf{M}_o\}$ and $\{\mathbf{K}_a, \mathbf{M}_a\}$. The vector of residuals \mathbf{y} can be written as

$$\mathbf{r} = \begin{bmatrix} \text{vec}(\mathbf{O}_K - \mathbf{A}) \\ \text{vec}(\mathbf{O}_M - \mathbf{A}) \end{bmatrix} \quad (8.44)$$

The sensitivity matrix of mutual orthogonality measures in equations (8.42) and (8.43) can be obtained either numerically or analytically. The derivatives of these measures are introduced in Chapter 7, section 7.3.5.

8.3 Model updating of generally damped systems

In this section, the general ideas for model updating the undamped systems in section 8.2 are extended to the case of damped systems. The model structure for generally damped systems remains as assumed in equations (8.1), (8.2) and (8.3). Model updating procedures are applied to modify the model parameters. The model parameters are adjusted so that output data from the model matches output data from the physical system. These sets of output data can be vectorised as before to form a vector of residuals \mathbf{r} .

8.3.1 Model updating based on eigenvalues and eigenvectors for general damped systems

8.3.1.1 Eigenvalues and eigenvectors for generally damped systems

The system dynamics of a general second order system are written as

$$\begin{bmatrix} \mathbf{K}_a & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_a \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} - \begin{bmatrix} -\mathbf{D}_a & -\mathbf{M}_a \\ -\mathbf{M}_a & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} \quad (8.45)$$

When no forcing is present, this becomes

$$\underline{\mathbf{D}}_a \underline{\dot{\mathbf{x}}} - \underline{\mathbf{K}}_a \underline{\dot{\mathbf{x}}} = \mathbf{0} \quad (8.46)$$

where $\underline{\mathbf{D}}_a$ and $\underline{\mathbf{K}}_a$ are $(2n \times 2n)$ matrices (see section 2.3.2 for further discussion on these). A generalised eigenvalue-eigenvector problem defined using matrices $\underline{\mathbf{D}}_a$ and $\underline{\mathbf{K}}_a$ above yields the triple of $(2n \times 2n)$ matrices, $\{\underline{\Phi}_{Ra}, \underline{\Lambda}_a, \underline{\Phi}_{La}\}$. The eigenvalues with their associated eigenvectors can be either real or complex conjugate which are presented by equations (2.27), (2.28) and (2.31)-(2.34).

$$\underline{\Phi}_{Ra} = \begin{bmatrix} \Phi_{Ra} \\ \Phi_{Ra} \Lambda_a \end{bmatrix} = \begin{bmatrix} \Phi_{Rca} & \Phi_{Rra1} & \Phi_{Rca}^* & \Phi_{Rra2} \\ \Phi_{Rca} \Lambda_{ca} & \Phi_{Rra1} \Lambda_{ra1} & \Phi_{Rca}^* \Lambda_{ca}^* & \Phi_{Rra2} \Lambda_{ra2} \end{bmatrix} \quad (8.47)$$

$$\underline{\Phi}_{La} = \begin{bmatrix} \Phi_{La} \\ \Phi_{La} \Lambda_a \end{bmatrix} = \begin{bmatrix} \Phi_{Lca} & \Phi_{Lra1} & \Phi_{Lca}^* & \Phi_{Lra2} \\ \Phi_{Lca} \Lambda_{ca} & \Phi_{Lra1} \Lambda_{ra1} & \Phi_{Lca}^* \Lambda_{ca}^* & \Phi_{Lra2} \Lambda_{ra2} \end{bmatrix} \quad (8.48)$$

The superscript notation “*” indicates a complex conjugate. Λ_a is (usually) a diagonal $(2n \times 2n)$ matrix and $\{\Phi_{Ra}, \Phi_{La}\}$ are $(n \times 2n)$ matrices obeying

$$\mathbf{M}_a \Phi_{Ra} \Lambda_a^2 + \mathbf{D}_a \Phi_{Ra} \Lambda_a + \mathbf{K}_a \Phi_{Ra} = \mathbf{0} \quad (8.49)$$

$$\mathbf{M}_a^T \Phi_{La} \Lambda_a^2 + \mathbf{D}_a^T \Phi_{La} \Lambda_a + \mathbf{K}_a^T \Phi_{La} = \mathbf{0} \quad (8.50)$$

The pairs of matrices (Φ_{Ra}, Λ_a) and (Φ_{La}, Λ_a) are called standard pairs for equation (8.45) if Φ_{Ra} and Φ_{La} from equations (8.47) and (8.48) are non-singular and equations (8.49) and (8.50) are obeyed [141, 142].

8.3.1.2 Scaling and ordering Eigenvectors for general damped systems

The measured eigenvectors are compared with the predicted eigenvectors according to some rules. This involves two steps:

Scaling the eigenvectors and

Ordering the eigenvectors of the analytical system to best match those of the measured system.

The right and left experimental and analytical modal matrices $\{\Phi_{Ro}, \Phi_{Lo}\}$, $\{\Phi_{Ra}, \Phi_{La}\}$ respectively, may be a mixture of real and complex conjugate. The real measured modal matrices $\{\Phi_{RV}, \Phi_{LV}\}$ are expressed in equation (2.38). These matrices have dimensions $(2n \times 2n)$.

$$\Phi_{RV} = \left[\begin{array}{cc|cc} (\Phi_{Rco} + \Phi_{Rco}^*) & \Phi_{Rro1} & (\Phi_{Rco} - \Phi_{Rco}^*) & \Phi_{Rro2} \\ \hline (\Phi_{Rco} \Lambda_{co} + \Phi_{Rco}^* \Lambda_{co}^*) & \Phi_{Rro1} \Lambda_{ro1} & (\Phi_{Rco} \Lambda_{co} - \Phi_{Rco}^* \Lambda_{co}^*) & \Phi_{Rro2} \Lambda_{ro2} \end{array} \right] \quad (8.51)$$

$$\Phi_{LV} = \left[\begin{array}{cc|cc} (\Phi_{Lco} + \Phi_{Lco}^*) & \Phi_{Lro1} & (\Phi_{Lco} - \Phi_{Lco}^*) & \Phi_{Lro2} \\ \hline (\Phi_{Lco} \Lambda_{co} + \Phi_{Lco}^* \Lambda_{co}^*) & \Phi_{Lro1} \Lambda_{ro1} & (\Phi_{Lco} \Lambda_{co} - \Phi_{Lco}^* \Lambda_{co}^*) & \Phi_{Lro2} \Lambda_{ro2} \end{array} \right] \quad (8.52)$$

and the real predicated modal matrices $\{\Phi_{RW}, \Phi_{LW}\}$ are expressed in equation (2.39).

See section 2.3 for further information. These matrices have dimensions $(2n \times 2n)$.

$$\Phi_{RW} = \left[\begin{array}{cc|cc} (\Phi_{Rca} + \Phi_{Rca}^*) & \Phi_{Rra1} & (\Phi_{Rca} - \Phi_{Rca}^*) & \Phi_{Rra2} \\ \hline (\Phi_{Rca} \Lambda_{ca} + \Phi_{Rca}^* \Lambda_{ca}^*) & \Phi_{Rra1} \Lambda_{ra1} & (\Phi_{Rca} \Lambda_{ca} - \Phi_{Rca}^* \Lambda_{ca}^*) & \Phi_{Rra2} \Lambda_{ra2} \end{array} \right] \quad (8.53)$$

$$\underline{\Phi}_{LW} = \left[\begin{array}{cc|cc} (\Phi_{Lca} + \Phi_{Lca}^*) & \Phi_{Lra1} & (\Phi_{Lca} - \Phi_{Lca}^*) & \Phi_{Lra2} \\ \hline (\Phi_{Lca} \Lambda_{ca} + \Phi_{Lca}^* \Lambda_{ca}^*) & \Phi_{Lra1} \Lambda_{ra1} & (\Phi_{Lca} \Lambda_{ca} - \Phi_{Lca}^* \Lambda_{ca}^*) & \Phi_{Lra2} \Lambda_{ra2} \end{array} \right] \quad (8.54)$$

The projection of subspace \mathbf{v}_{Ri} of the modal matrix $\underline{\Phi}_{RV}$ onto the subspace \mathbf{w}_{Ri} of modal matrix $\underline{\Phi}_{RW}$ can be defined by \mathbf{z}_{Rij} . The pair of eigenvectors \mathbf{v}_{Ri} defines the i^{th} and $i^{\text{th}}+n$ columns of the modal matrix $\underline{\Phi}_{RV}$ of the measured system where ($i=1,2,\dots,n$). The pair of eigenvectors \mathbf{w}_{Rj} defines the j^{th} and $j^{\text{th}}+n$ columns in the modal matrix $\underline{\Phi}_{RW}$ of the analytical system where ($j=1,2,\dots,n$). The dimensions of \mathbf{v}_{Ri} and \mathbf{w}_{Rj} are $(2n \times 2)$. The pair of columns \mathbf{z}_{Rij} is the projection of \mathbf{v}_{Ri} onto \mathbf{w}_{Rj} . The projection \mathbf{z}_{Rij} has dimension $(2n \times 2)$ and is defined as

$$\mathbf{z}_{Rij} = \mathbf{P} \mathbf{v}_{Ri} = \mathbf{w}_{Rj} \left(\mathbf{w}_{Rj}^T \mathbf{w}_{Rj} \right)^{-1} \mathbf{w}_{Rj}^T \mathbf{v}_{Ri} \quad (8.55)$$

where \mathbf{P} is a projection matrix (satisfying an idempotent $\mathbf{P}^2 = \mathbf{P} = \mathbf{P}^3 = \dots$).

Figure 8-1 shows the projection of the subspace \mathbf{v}_{Ri} onto the subspace \mathbf{w}_{Rj} . If the projection \mathbf{z}_{Rij} lays within the subspace \mathbf{w}_{Rj} , then the angle between the two subspaces $\{\mathbf{z}_{Rij}, \mathbf{v}_{Ri}\}$ must be zero. The cosine of the angle α between \mathbf{z}_{Rij} and \mathbf{v}_{Ri} can be expressed as

$$x_{Rij} = \cos(\alpha) = \frac{\left(\text{norm}(\mathbf{z}_{Rij}^T \mathbf{v}_{Ri}) \right)^2}{\left(\text{norm}(\mathbf{v}_{Ri}^T \mathbf{v}_{Ri}) \right)^2} = \frac{\left\| \mathbf{z}_{Rij}^T \mathbf{v}_{Ri} \right\|}{\left\| \mathbf{v}_{Ri}^T \mathbf{v}_{Ri} \right\|} \quad (8.56)$$

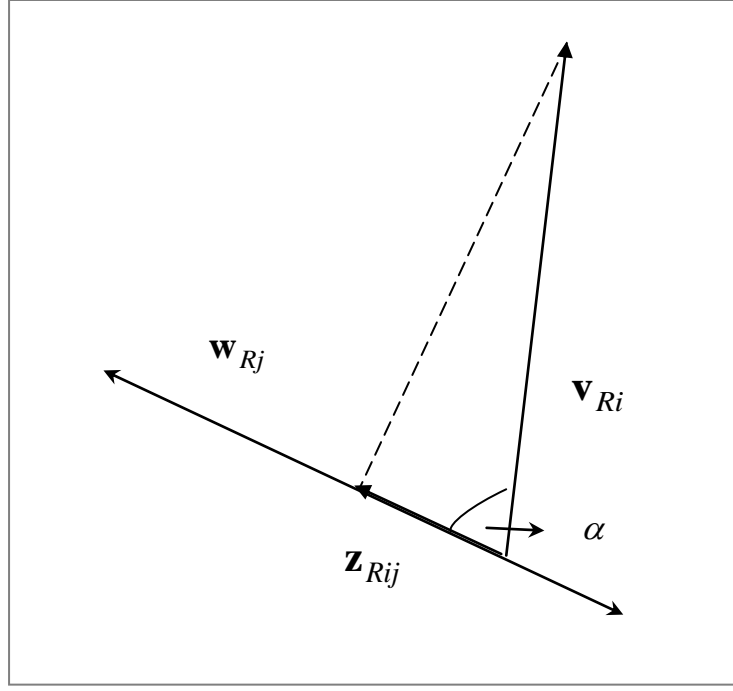


Figure 8-1 : Projection of \mathbf{v}_{Ri} into \mathbf{w}_{Rj}

The notation $\|\cdot\|$ represents the square of the norm. Each pair of eigenvectors \mathbf{v}_{Ri} of Φ_{RV} (where $i=1,2,\dots,n$) is compared with the pair of eigenvectors \mathbf{w}_{Rj} of Φ_{RW} (and $j=1,2,\dots,n$), where all values of j are spanned for a given i . Thus, a given instance of \mathbf{v}_{Ri} is matched with a total n pairs of eigenvectors from Φ_{RW} , and the result in each case forms the $(i,j)^{\text{th}}$ entry of the matrix \mathfrak{X}_R .

$$\mathfrak{X}_R = \begin{bmatrix} x_{R11} & \cdots & x_{R1j} \\ \vdots & \cdots & \vdots \\ x_{Ri1} & \cdots & x_{Rij} \end{bmatrix} \quad (8.57)$$

The matrix \mathfrak{X}_R is $(n \times n)$ and this represents the cosine of the angles between each subspace in Φ_{RV} with the subspace in Φ_{RW} . The location of the largest value of x_{Rij} in each row of matrix \mathfrak{X}_R indicates which subspace \mathbf{v}_{Ri} matches most closely to subspace \mathbf{w}_{Rj} .

$$[\mathfrak{g}_R \quad \mathfrak{L}_R] = \max(\mathfrak{X}_R^T) \quad (8.58)$$

The \mathfrak{g}_R has dimension $(1 \times n)$, the entries indicate the maximum values in the rows of the matrix \mathfrak{X}_R . The \mathfrak{L}_R has dimension $(1 \times n)$ and the entries indicate the location of the largest values \mathfrak{g}_R of the rows in \mathfrak{X}_R . The n pairs of eigenvectors in the matrix $\underline{\Phi}_{RW}$ can be ordered related to the location \mathfrak{L}_R of the largest values in the rows in \mathfrak{X}_R .

$$\underline{\Phi}_{RW} = [\underline{\Phi}_{RW}(:, \mathfrak{L}_R) \quad \underline{\Phi}_{RW}(:, (\mathfrak{L}_R + n))] \quad (8.59)$$

The same procedure is used to match and order the left real predicted modal matrix $\underline{\Phi}_{LW}$ related to the left real measured modal matrix $\underline{\Phi}_{LV}$.

8.3.1.3 Residual measures based on eigenvectors for general damped systems

Vector of residuals \mathbf{y} has dimension $(8n^2)$. The vector of residuals for the right eigenvectors can be written as

$$\mathbf{r}_R = \left[\text{vec}(\mathbf{v}_{Ri} - \mathbf{z}_{Ri}) \right]_{i=1}^n = \begin{bmatrix} \text{vec} \left(\mathbf{v}_{R1} - \left(\mathbf{w}_{R1} (\mathbf{w}_{R1}^T \mathbf{w}_{R1})^{-1} (\mathbf{w}_{R1}^T \mathbf{v}_{R1}) \right) \right) \\ \text{vec} \left(\mathbf{v}_{R2} - \left(\mathbf{w}_{R2} (\mathbf{w}_{R2}^T \mathbf{w}_{R2})^{-1} (\mathbf{w}_{R2}^T \mathbf{v}_{R2}) \right) \right) \\ \vdots \\ \text{vec} \left(\mathbf{v}_{Rn} - \left(\mathbf{w}_{Rn} (\mathbf{w}_{Rn}^T \mathbf{w}_{Rn})^{-1} (\mathbf{w}_{Rn}^T \mathbf{v}_{Rn}) \right) \right) \end{bmatrix} \quad (8.60)$$

The vector of residuals for the left eigenvectors can be written as

$$\mathbf{r}_L = \left[\text{vec}(\mathbf{v}_{Li} - \mathbf{z}_{Li}) \right]_{i=1}^n = \begin{bmatrix} \text{vec} \left(\mathbf{v}_{L1} - \left(\mathbf{w}_{L1} (\mathbf{w}_{L1}^T \mathbf{w}_{L1})^{-1} (\mathbf{w}_{L1}^T \mathbf{v}_{L1}) \right) \right) \\ \text{vec} \left(\mathbf{v}_{L2} - \left(\mathbf{w}_{L2} (\mathbf{w}_{L2}^T \mathbf{w}_{L2})^{-1} (\mathbf{w}_{L2}^T \mathbf{v}_{L2}) \right) \right) \\ \vdots \\ \text{vec} \left(\mathbf{v}_{Ln} - \left(\mathbf{w}_{Ln} (\mathbf{w}_{Ln}^T \mathbf{w}_{Ln})^{-1} (\mathbf{w}_{Ln}^T \mathbf{v}_{Ln}) \right) \right) \end{bmatrix} \quad (8.61)$$

Model updating may be attempted using the above two vectors of residuals $\{\mathbf{r}_R, \mathbf{r}_L\}$ which can be combined into one vector giving

$$\mathbf{r} = \begin{bmatrix} \mathbf{r}_R \\ \mathbf{r}_L \end{bmatrix} \quad (8.62)$$

The sensitivity matrix in equation (8.8) can be computed analytically by calculating the derivatives of eigenvalues and eigenvectors with respect to each single scalar parameter in vector $\boldsymbol{\theta}$. The rates of change of eigenvalues and eigenvectors for the damped system in equations (8.1), (8.2) and (8.3) are developed in Chapter 5 (and more details are in Appendix C).

8.3.2 *Model Updating based on diagonalising SPEs*

Since the system matrices $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$ are changed, the corresponding diagonalised system matrices and the diagonalising transformations are also changed. The right and left diagonalising transformations $\{\underline{\mathbf{T}}_{Ra}, \underline{\mathbf{T}}_{La}\}$ for the analytical system are functions of $\boldsymbol{\theta}$. We assume that $\Delta\boldsymbol{\theta}$ are small.

$$\underline{\mathbf{T}}_{Ra} = \underline{\mathbf{T}}_{Ro} + \Delta \underline{\mathbf{T}}_R = \underline{\mathbf{T}}_{Ro} + \sum_{j=1}^p \frac{\partial \underline{\mathbf{T}}_{Ra}}{\partial \theta_j} \Delta \theta_j \quad (8.63)$$

$$\underline{\mathbf{T}}_{La} = \underline{\mathbf{T}}_{Lo} + \Delta \underline{\mathbf{T}}_L = \underline{\mathbf{T}}_{Lo} + \sum_{j=1}^p \frac{\partial \underline{\mathbf{T}}_{La}}{\partial \theta_j} \Delta \theta_j \quad (8.64)$$

Left and right diagonalising SPEs $\{\underline{\mathbf{T}}_{Ro}, \underline{\mathbf{T}}_{Lo}\}$ for the physical (true) system can be found to diagonalise the system matrices $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$. The diagonalising SPEs $\{\underline{\mathbf{T}}_{Ra}, \underline{\mathbf{T}}_{La}\}$ diagonalise the analytical system $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$. The right and left diagonalising transformations $\{\underline{\mathbf{T}}_{Ro}, \underline{\mathbf{T}}_{Lo}\}$ for systems $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ satisfy the following equations,

$$\begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix}^T \begin{bmatrix} 0 & \mathbf{K}_o \\ \mathbf{K}_o & \mathbf{D}_o \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{K}_{Do} \\ \mathbf{K}_{Do} & \mathbf{D}_{Do} \end{bmatrix} \quad (8.65)$$

$$\begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_o & 0 \\ 0 & -\mathbf{M}_o \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{Do} & 0 \\ 0 & -\mathbf{M}_{Do} \end{bmatrix} \quad (8.66)$$

$$\begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix}^T \begin{bmatrix} -\mathbf{D}_o & -\mathbf{M}_o \\ -\mathbf{M}_o & 0 \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} = \begin{bmatrix} -\mathbf{D}_{Do} & -\mathbf{M}_{Do} \\ -\mathbf{M}_{Do} & 0 \end{bmatrix} \quad (8.67)$$

The right and left diagonalising transformations $\{\underline{\mathbf{T}}_{Ra}, \underline{\mathbf{T}}_{La}\}$ for systems $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$ satisfy the following equations,

$$\begin{bmatrix} \mathbf{W}_{La} & \mathbf{X}_{La} \\ \mathbf{Y}_{La} & \mathbf{Z}_{La} \end{bmatrix}^T \begin{bmatrix} 0 & \mathbf{K}_a \\ \mathbf{K}_a & \mathbf{D}_a \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ra} & \mathbf{X}_{Ra} \\ \mathbf{Y}_{Ra} & \mathbf{Z}_{Ra} \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{K}_{Da} \\ \mathbf{K}_{Da} & \mathbf{D}_{Da} \end{bmatrix} \quad (8.68)$$

$$\begin{bmatrix} \mathbf{W}_{La} & \mathbf{X}_{La} \\ \mathbf{Y}_{La} & \mathbf{Z}_{La} \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_a & 0 \\ 0 & -\mathbf{M}_a \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ra} & \mathbf{X}_{Ra} \\ \mathbf{Y}_{Ra} & \mathbf{Z}_{Ra} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{Da} & 0 \\ 0 & -\mathbf{M}_{Da} \end{bmatrix} \quad (8.69)$$

$$\begin{bmatrix} \mathbf{W}_{La} & \mathbf{X}_{La} \\ \mathbf{Y}_{La} & \mathbf{Z}_{La} \end{bmatrix}^T \begin{bmatrix} -\mathbf{D}_a & -\mathbf{M}_a \\ -\mathbf{M}_a & 0 \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ra} & \mathbf{X}_{Ra} \\ \mathbf{Y}_{Ra} & \mathbf{Z}_{Ra} \end{bmatrix} = \begin{bmatrix} -\mathbf{D}_{Da} & -\mathbf{M}_{Da} \\ -\mathbf{M}_{Da} & 0 \end{bmatrix} \quad (8.70)$$

The new model updating methods proposed here also use the discrepancy between computed and measured results and they use sensitivities to determine a change in the update parameters that will reduce the discrepancy. The right and left diagonalising transformations are used to update the model parameters. The derivatives of the diagonalised system matrices and their diagonalising

transformations are introduced in Chapter 5. The model parameters are iteratively adjusted until the numerical (or analytical) models match the measured models.

Two problems are noted: the diagonalising transformations are not unique because of (a) ordering and (b) scaling.

8.3.2.1 *Scaling and ordering diagonalising SPEs*

In this section, the ideas for undamped systems in section 8.2.1.3 are extended for generally damped system. This section attempts to find the best "ordering" of pairs of vectors within diagonalising transformations from the system $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$ and "scaling" of those vectors. The resulting ordered and scaled vectors are "like" the diagonalising transformations of the system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$. Arranging the eigenvalues in ascending order of magnitude is not sufficient, especially when two modes are close together in frequency.

The experimental and theoretical eigenvalues and eigenvectors must relate to the same mode. Usually the analytical eigenvectors are mass normalised and the measured eigenvectors obtained from standard modal analysis methods are also mass normalised. Because the mass distributions of the finite element model and the actual structure may be different, the eigenvectors may not be scaled consistently. Without loss of generality the diagonalising SPEs for both systems can be selected such that

$$k_i^2 + d_i^2 + m_i^2 = 1 \quad \forall \quad 0 < i < n \quad (8.71)$$

in which $\{k_i, d_i, m_i\}$ are the i^{th} entries of the diagonalised system matrices $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$. Each triple of scalars $\{k_i, d_i, m_i\}$ defines a pair of eigenvalues implicitly through

$$k_i + \lambda d_i + \lambda^2 m_i = 0 \quad (8.72)$$

The linear combination of equations (8.65)-(8.67) and linear combination of equations (8.68)-(8.70) that give at least two zero singular values. In this case, non-eigenvalues are chosen for which no singular value is close to zero and to give a well conditioned matrix. The linear combination of the equations which have been mentioned above can be written as

$$\underline{\mathbf{X}}_o = \begin{pmatrix} \alpha \begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix}^T \begin{bmatrix} 0 & \mathbf{K}_a \\ \mathbf{K}_a & \mathbf{D}_a \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} + \beta \begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_a & 0 \\ 0 & -\mathbf{M}_a \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} \\ + \gamma \begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix}^T \begin{bmatrix} -\mathbf{D}_a & -\mathbf{M}_a \\ -\mathbf{M}_a & 0 \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} \end{pmatrix} \quad (8.73)$$

$$\underline{\mathbf{X}}_a = \begin{pmatrix} \alpha \begin{bmatrix} \mathbf{W}_{La} & \mathbf{X}_{La} \\ \mathbf{Y}_{La} & \mathbf{Z}_{La} \end{bmatrix}^T \begin{bmatrix} 0 & \mathbf{K}_a \\ \mathbf{K}_a & \mathbf{D}_a \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ra} & \mathbf{X}_{Ra} \\ \mathbf{Y}_{Ra} & \mathbf{Z}_{Ra} \end{bmatrix} + \beta \begin{bmatrix} \mathbf{W}_{La} & \mathbf{X}_{La} \\ \mathbf{Y}_{La} & \mathbf{Z}_{La} \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_a & 0 \\ 0 & -\mathbf{M}_a \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ra} & \mathbf{X}_{Ra} \\ \mathbf{Y}_{Ra} & \mathbf{Z}_{Ra} \end{bmatrix} \\ + \gamma \begin{bmatrix} \mathbf{W}_{La} & \mathbf{X}_{La} \\ \mathbf{Y}_{La} & \mathbf{Z}_{La} \end{bmatrix}^T \begin{bmatrix} -\mathbf{D}_a & -\mathbf{M}_a \\ -\mathbf{M}_a & 0 \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ra} & \mathbf{X}_{Ra} \\ \mathbf{Y}_{Ra} & \mathbf{Z}_{Ra} \end{bmatrix} \end{pmatrix} \quad (8.74)$$

In which the best values of $\{\alpha, \beta, \gamma\}$ are chosen to be

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \quad (8.75)$$

and at minimum condition for the matrix

$$\min \left(\text{cond} \left(\alpha \underline{\mathbf{M}}_a + \beta \underline{\mathbf{D}}_a + \gamma \underline{\mathbf{K}}_a \right) \right) \quad (8.76)$$

The entries in the set $\{\alpha, \beta, \gamma\}$ are called *negievalues*.

$$\alpha + \lambda_i \beta + \lambda_i^2 \gamma \neq 0 \quad (8.77)$$

The right and left diagonalising SPEs for both systems $\{\underline{\mathbf{T}}_{Rso}, \underline{\mathbf{T}}_{Lso}\}$, $\{\underline{\mathbf{T}}_{Rsa}, \underline{\mathbf{T}}_{Lsa}\}$ are scaled according to the above conditions in equations (8.75), (8.76) and (8.77).

Rewrite equations (8.73) and (8.74) with scaled diagonalising SPEs $\{\underline{\mathbf{T}}_{Rso}, \underline{\mathbf{T}}_{Lso}\}$,

$$\{\underline{\mathbf{T}}_{Rsa}, \underline{\mathbf{T}}_{Lsa}\}$$

$$\underline{\mathbf{X}}_{so} = \left(\alpha \underline{\mathbf{T}}_{Lso}^T \underline{\mathbf{M}}_a \underline{\mathbf{T}}_{Rso} + \beta \underline{\mathbf{T}}_{Lso}^T \underline{\mathbf{D}}_a \underline{\mathbf{T}}_{Rso} + \gamma \underline{\mathbf{T}}_{Lso}^T \underline{\mathbf{K}}_a \underline{\mathbf{T}}_{Rso} \right) \quad (8.78)$$

$$\underline{\mathbf{X}}_{sa} = \left(\alpha \underline{\mathbf{T}}_{Lsa}^T \underline{\mathbf{M}}_a \underline{\mathbf{T}}_{Rsa} + \beta \underline{\mathbf{T}}_{Lsa}^T \underline{\mathbf{D}}_a \underline{\mathbf{T}}_{Rsa} + \gamma \underline{\mathbf{T}}_{Lsa}^T \underline{\mathbf{K}}_a \underline{\mathbf{T}}_{Rsa} \right) \quad (8.79)$$

Balancing the scaled diagonalising SPEs of system $\{\underline{\mathbf{K}}_a, \underline{\mathbf{D}}_a, \underline{\mathbf{M}}_a\}$

$$\underline{\mathbf{t}}_o = \underline{\mathbf{X}}_{so} \underline{\mathbf{X}}_{so}^T, \quad \underline{\mathbf{t}}_a = \underline{\mathbf{X}}_{sa} \underline{\mathbf{X}}_{sa}^T \quad (8.80)$$

The matrices $\{\underline{\mathbf{t}}_o, \underline{\mathbf{t}}_a\}$ have four blocks each. The sum of the diagonal entries of each

block can be written as

$$\underline{\mathbf{t}}_o = \begin{pmatrix} \text{diag}(\underline{\mathbf{t}}_{o11}) + \text{diag}(\underline{\mathbf{t}}_{o22}) + \\ (\text{diag}(\underline{\mathbf{t}}_{o12}) + \text{diag}(\underline{\mathbf{t}}_{o21})) / 2 \end{pmatrix}, \quad \underline{\mathbf{t}}_a = \begin{pmatrix} \text{diag}(\underline{\mathbf{t}}_{a11}) + \text{diag}(\underline{\mathbf{t}}_{a22}) + \\ (\text{diag}(\underline{\mathbf{t}}_{a12}) + \text{diag}(\underline{\mathbf{t}}_{a21})) / 2 \end{pmatrix}$$

$$\underline{\mathbf{t}}_b = \text{diag} \left(\sqrt{\sqrt{(\underline{\mathbf{t}}_o \cdot \underline{\mathbf{t}}_a)}} \right), \quad \underline{\mathbf{t}}_b = \begin{bmatrix} \underline{\mathbf{t}}_b & 0 \\ 0 & \underline{\mathbf{t}}_b \end{bmatrix} \quad (8.81)$$

where $\underline{\mathbf{t}}_b$ is a matrix which has a dimension $(2n \times 2n)$.

The scaled right and left diagonalising SPEs $\{\underline{\mathbf{T}}_{Rsa}, \underline{\mathbf{T}}_{Lsa}\}$ and the matrices $\{\underline{\mathbf{X}}_{so}, \underline{\mathbf{X}}_{sa}\}$

are balanced two columns at a time which can be written as

$$\underline{\mathbf{T}}_{Rba} = \underline{\mathbf{T}}_{Rsa} / \underline{\mathbf{t}}_b, \quad \underline{\mathbf{T}}_{Lba} = \underline{\mathbf{T}}_{Lsa} \underline{\mathbf{t}}_b \quad (8.82)$$

$$\underline{\mathbf{X}}_{bo} = \underline{\mathbf{X}}_{so} / \underline{\mathbf{t}}_b, \quad \underline{\mathbf{X}}_{ba} = \underline{\mathbf{X}}_{sa} \underline{\mathbf{t}}_b \quad (8.83)$$

The sum of the matrices $\{\underline{\mathbf{X}}_{bo}, \underline{\mathbf{X}}_{ba}\}$ can be written as

$$\underline{\mathbf{X}} = \underline{\mathbf{X}}_{bo} + \underline{\mathbf{X}}_{ba} \quad (8.84)$$

To match the computed diagonalising SPEs $\{\underline{\mathbf{T}}_{Ra}, \underline{\mathbf{T}}_{La}\}$ with measured diagonalising SPEs $\{\underline{\mathbf{T}}_{Ro}, \underline{\mathbf{T}}_{Lo}\}$, the right and left diagonalising SPEs $\{\underline{\mathbf{T}}_{Ra}, \underline{\mathbf{T}}_{La}\}$ are ordered two columns at a time according to the locations of the largest values in the rows of the matrix $\underline{\mathbf{X}}$.

$$[\mathcal{I}_{ii} \quad \mathfrak{L}_{ii}] = \max(\underline{\mathbf{X}}(ii, irng)) \quad ii = 1, 2, \dots, n \quad (8.85)$$

$$irng = [(ii : n) \quad (ii + n : 2n)]$$

$$\underline{\mathbf{T}}_{Ra} = [\underline{\mathbf{T}}_{Rba}(:, \mathfrak{L}_{ii}) \quad \underline{\mathbf{T}}_{Rba}(:, \mathfrak{L}_{ii} + n)] \quad , \quad \underline{\mathbf{T}}_{La} = [\underline{\mathbf{T}}_{Lba}(:, \mathfrak{L}_{ii}) \quad \underline{\mathbf{T}}_{Lba}(:, \mathfrak{L}_{ii} + n)] \quad (8.86)$$

8.3.2.2 Residual measures based on diagonalising SPEs

The vector of residuals \mathbf{r} based on the diagonalising SPEs is real and has dimension $(8n^2)$.

$$\mathbf{r} = \begin{bmatrix} \text{vec}(\underline{\mathbf{T}}_{Ro} - \underline{\mathbf{T}}_{Ra}) \\ \text{vec}(\underline{\mathbf{T}}_{Lo} - \underline{\mathbf{T}}_{La}) \end{bmatrix} \quad (8.87)$$

The derivatives of diagonalising SPEs are developed and introduced in Chapter 5. The sensitivity matrix expressing the derivatives of diagonalising SPEs with respect to parameter vector $\boldsymbol{\theta}$ can be computed analytically.

Model updating based on diagonalising SPEs requires complete modal information for both systems $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ and $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$. The right and left diagonalising SPEs for the analytical system $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$ must be in the same order with the right and left diagonalising SPEs for the measured system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$.

8.3.3 Model updating based on homogeneous coordinates

8.3.3.1 Linear Combination of mutual orthogonality checks

Linear combination of *mutual orthogonality checks* of equations (8.65), (8.66) and (8.67) for system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ are shown

$$\begin{aligned} & \left(\begin{bmatrix} \mathbf{0} & \mathbf{K}_o \\ \mathbf{K}_o & \mathbf{D}_o \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} \begin{bmatrix} \mathbf{M}_{Do} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{Do} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_o & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_o \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} \begin{bmatrix} \mathbf{D}_{Do} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{Do} \end{bmatrix} \right. \\ & \left. + \begin{bmatrix} -\mathbf{D}_o & -\mathbf{M}_o \\ -\mathbf{M}_o & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} \begin{bmatrix} \mathbf{K}_{Do} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{Do} \end{bmatrix} \right) = \\ & \begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix}^{-T} \left(\begin{bmatrix} \mathbf{0} & \mathbf{K}_{Do} \\ \mathbf{K}_{Do} & \mathbf{D}_{Do} \end{bmatrix} \begin{bmatrix} \mathbf{M}_{Do} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{Do} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{Do} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{Do} \end{bmatrix} \begin{bmatrix} \mathbf{D}_{Do} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{Do} \end{bmatrix} \right. \\ & \left. + \begin{bmatrix} -\mathbf{D}_{Do} & -\mathbf{M}_{Do} \\ -\mathbf{M}_{Do} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{K}_{Do} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{Do} \end{bmatrix} \right) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \end{aligned} \quad (8.88)$$

Linear combination of *mutual orthogonality checks* of equations (8.68), (8.69) and (8.70) for system $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$ are shown

$$\begin{aligned} & \left(\begin{bmatrix} \mathbf{0} & \mathbf{K}_a \\ \mathbf{K}_a & \mathbf{D}_a \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ra} & \mathbf{X}_{Ra} \\ \mathbf{Y}_{Ra} & \mathbf{Z}_{Ra} \end{bmatrix} \begin{bmatrix} \mathbf{M}_{Da} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{Da} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_a & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_a \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ra} & \mathbf{X}_{Ra} \\ \mathbf{Y}_{Ra} & \mathbf{Z}_{Ra} \end{bmatrix} \begin{bmatrix} \mathbf{D}_{Da} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{Da} \end{bmatrix} \right) = \\ & \left(\begin{bmatrix} -\mathbf{D}_a & -\mathbf{M}_a \\ -\mathbf{M}_a & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ra} & \mathbf{X}_{Ra} \\ \mathbf{Y}_{Ra} & \mathbf{Z}_{Ra} \end{bmatrix} \begin{bmatrix} \mathbf{K}_{Da} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{Da} \end{bmatrix} \right) = \\ & \begin{bmatrix} \mathbf{W}_{La} & \mathbf{X}_{La} \\ \mathbf{Y}_{La} & \mathbf{Z}_{La} \end{bmatrix}^{-T} \left(\begin{bmatrix} \mathbf{0} & \mathbf{K}_{Da} \\ \mathbf{K}_{Da} & \mathbf{D}_{Da} \end{bmatrix} \begin{bmatrix} \mathbf{M}_{Da} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{Da} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{Da} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{Da} \end{bmatrix} \begin{bmatrix} \mathbf{D}_{Da} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{Da} \end{bmatrix} \right. \\ & \left. + \begin{bmatrix} -\mathbf{D}_{Da} & -\mathbf{M}_{Da} \\ -\mathbf{M}_{Da} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{K}_{Da} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{Da} \end{bmatrix} \right) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \end{aligned} \quad (8.89)$$

8.3.3.2 Linear combination of Cross orthogonality checks

Cross orthogonality checks between the numerical and the measured systems when the left modal information $\underline{\mathbf{T}}_{Lo}$ for the physical system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ is known are defined below.

$$\underline{\mathbf{X}}_{LMoa} := \begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix}^T \begin{bmatrix} 0 & \mathbf{K}_a \\ \mathbf{K}_a & \mathbf{D}_a \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ra} & \mathbf{X}_{Ra} \\ \mathbf{Y}_{Ra} & \mathbf{Z}_{Ra} \end{bmatrix} \quad (8.90)$$

$$\underline{\mathbf{X}}_{LDoa} := \begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_a & 0 \\ 0 & -\mathbf{M}_a \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ra} & \mathbf{X}_{Ra} \\ \mathbf{Y}_{Ra} & \mathbf{Z}_{Ra} \end{bmatrix} \quad (8.91)$$

$$\underline{\mathbf{X}}_{LKoa} := \begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix}^T \begin{bmatrix} -\mathbf{D}_a & -\mathbf{M}_a \\ -\mathbf{M}_a & 0 \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ra} & \mathbf{X}_{Ra} \\ \mathbf{Y}_{Ra} & \mathbf{Z}_{Ra} \end{bmatrix} \quad (8.92)$$

If the system matrices $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$ and $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ are identical, and the diagonalising SPE $\underline{\mathbf{T}}_{La}$ matches the DSPE $\underline{\mathbf{T}}_{Lo}$. Then the matrices $\underline{\mathbf{X}}_{LMoa}$, $\underline{\mathbf{X}}_{LDoa}$ and $\underline{\mathbf{X}}_{LKoa}$ are block diagonal which are similar to equations (8.65)-(8.67).

Pre-multiplying equations (8.90), (8.91) and (8.92) by \mathbf{M}_{Do} , \mathbf{D}_{Do} , \mathbf{K}_{Do} , respectively, and post-multiplying by the inverse of the right DSPEs $\underline{\mathbf{T}}_{Ra}$ for the system $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$ and adding the results of each yields

$$\begin{aligned} & \left(\begin{bmatrix} \mathbf{M}_{Do} & 0 \\ 0 & \mathbf{M}_{Do} \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix}^T \begin{bmatrix} 0 & \mathbf{K}_a \\ \mathbf{K}_a & \mathbf{D}_a \end{bmatrix} + \begin{bmatrix} \mathbf{D}_{Do} & 0 \\ 0 & \mathbf{D}_{Do} \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_a & 0 \\ 0 & -\mathbf{M}_a \end{bmatrix} + \right. \\ & \left. \begin{bmatrix} \mathbf{K}_{Do} & 0 \\ 0 & \mathbf{K}_{Do} \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix}^T \begin{bmatrix} -\mathbf{D}_a & -\mathbf{M}_a \\ -\mathbf{M}_a & 0 \end{bmatrix} \right) = \\ & \left(\begin{bmatrix} \mathbf{M}_{Do} & 0 \\ 0 & \mathbf{M}_{Do} \end{bmatrix} \underline{\mathbf{X}}_{LMoa} + \begin{bmatrix} \mathbf{D}_{Do} & 0 \\ 0 & \mathbf{D}_{Do} \end{bmatrix} \underline{\mathbf{X}}_{LDoa} + \begin{bmatrix} \mathbf{K}_{Do} & 0 \\ 0 & \mathbf{K}_{Do} \end{bmatrix} \underline{\mathbf{X}}_{LKoa} \right) \begin{bmatrix} \mathbf{W}_{Ra} & \mathbf{X}_{Ra} \\ \mathbf{Y}_{Ra} & \mathbf{Z}_{Ra} \end{bmatrix}^{-1} \end{aligned} \quad (8.93)$$

If the system matrices $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$ and the system matrices $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ are identical then the left hand side of equation (8.93) becomes zero $(2N \times 2N)$ matrix. Then, this equation can be written as

$$\begin{aligned} & \left(\begin{bmatrix} \mathbf{M}_{Do} & 0 \\ 0 & \mathbf{M}_{Do} \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix}^T \begin{bmatrix} 0 & \mathbf{K}_a \\ \mathbf{K}_a & \mathbf{D}_a \end{bmatrix} + \begin{bmatrix} \mathbf{D}_{Do} & 0 \\ 0 & \mathbf{D}_{Do} \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_a & 0 \\ 0 & -\mathbf{M}_a \end{bmatrix} + \right. \\ & \left. + \begin{bmatrix} \mathbf{K}_{Do} & 0 \\ 0 & \mathbf{K}_{Do} \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix}^T \begin{bmatrix} -\mathbf{D}_a & -\mathbf{M}_a \\ -\mathbf{M}_a & 0 \end{bmatrix} \right) \begin{bmatrix} \mathbf{W}_{Ra} & \mathbf{X}_{Ra} \\ \mathbf{Y}_{Ra} & \mathbf{Z}_{Ra} \end{bmatrix}^{-1} \end{aligned} \equiv \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (8.94)$$

The same process can be done when the right modes information $\underline{\mathbf{T}}_{Ro}$ have been measured. Then the cross orthogonality checks can be expressed as

$$\underline{\mathbf{X}}_{RMao} := \begin{bmatrix} \mathbf{W}_{La} & \mathbf{X}_{La} \\ \mathbf{Y}_{La} & \mathbf{Z}_{La} \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K}_a \\ \mathbf{K}_a & \mathbf{D}_a \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} \quad (8.95)$$

$$\underline{\mathbf{X}}_{RDao} := \begin{bmatrix} \mathbf{W}_{La} & \mathbf{X}_{La} \\ \mathbf{Y}_{La} & \mathbf{Z}_{La} \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_a & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_a \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} \quad (8.96)$$

$$\underline{\mathbf{X}}_{RKao} := \begin{bmatrix} \mathbf{W}_{La} & \mathbf{X}_{La} \\ \mathbf{Y}_{La} & \mathbf{Z}_{La} \end{bmatrix}^T \begin{bmatrix} -\mathbf{D}_a & -\mathbf{M}_a \\ -\mathbf{M}_a & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} \quad (8.97)$$

If the system matrices $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$ and the system matrices $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ are identical, the linear combination of *LAMs* for the system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ when right mode information $\underline{\mathbf{T}}_{Ro}$ is known is defined as

$$\left(\begin{bmatrix} \mathbf{0} & \mathbf{K}_a \\ \mathbf{K}_a & \mathbf{D}_a \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} \begin{bmatrix} \mathbf{M}_{Do} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{Do} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_a & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_a \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} \begin{bmatrix} \mathbf{D}_{Do} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{Do} \end{bmatrix} + \begin{bmatrix} -\mathbf{D}_a & -\mathbf{M}_a \\ -\mathbf{M}_a & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} \begin{bmatrix} \mathbf{K}_{Do} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{Do} \end{bmatrix} \right) \cong \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (8.98)$$

8.3.3.3 Updating Analytical models

The model updating method proposed is based on minimising discrepancies between computed and measured results as described above. A matrix of sensitivities is found to determine the changes in the update parameters that will reduce the discrepancy. The model parameters are iteratively adjusted until the predicted models match the measured models as well as possible. Both equations (8.94) and (8.98) are dependent on the measured modal information $\{\underline{\mathbf{T}}_{Lo}, \underline{\mathbf{T}}_{Ro}\}$, and numerical model matrices $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$.

8.3.3.4 Residual measures based on Linear Combination of LAMs

The measured modal information $\underline{\mathbf{T}}_{Ro}$ is used as a reference value and the analytical system matrices $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$ are functions of the unknown parameters Θ to be updated, the error is represented in the residual vector.

$$\mathbf{r} = \text{vec} \left(\begin{bmatrix} 0 & \mathbf{K}_a \\ \mathbf{K}_a & \mathbf{D}_a \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} \begin{bmatrix} \mathbf{M}_{Do} & 0 \\ 0 & \mathbf{M}_{Do} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_a & 0 \\ 0 & -\mathbf{M}_a \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} \begin{bmatrix} \mathbf{D}_{Do} & 0 \\ 0 & \mathbf{D}_{Do} \end{bmatrix} + \begin{bmatrix} -\mathbf{D}_a & -\mathbf{M}_a \\ -\mathbf{M}_a & 0 \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} \begin{bmatrix} \mathbf{K}_{Do} & 0 \\ 0 & \mathbf{K}_{Do} \end{bmatrix} \right) \quad (8.99)$$

This is only one side (the right side) of measure. The vector of residuals has dimensions $(4n^2)$. The vector of residuals may comprise the left side measure using equation (8.94) or both right and left of measured modes equations (8.94) and (8.98).

When applying the method of weighted least squares the minimisation of the objective function equation (8.99) permits the estimate of the correction parameter vector Θ where Θ is a vector parametering the uncertainties from the analytical models.

$$E = \mathbf{r}^T \mathbf{W} \mathbf{r} \quad (8.100)$$

The sensitivity matrix here is rank deficient (rank is less than number of unknowns) and has some zero singular values indicating that the number of variable parameters is more than the number of equations (underdetermined).

8.3.3.5 Improved Analytical Models

The updated systems matrices $\{\mathbf{K}_{up}, \mathbf{D}_{up}, \mathbf{M}_{up}\}$ satisfy equation (8.98) and also the measured system matrices $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ satisfy equation (8.88). Equation (8.98)

shows that using the right DSPE $\underline{\mathbf{T}}_{Ro}$ and simply scaling every entry of the system matrices $\{\mathbf{K}_{up}, \mathbf{D}_{up}, \mathbf{M}_{up}\}$ by any number does not make any change to that equation. Comparing these two equations for the same right measured diagonalising SPE $\underline{\mathbf{T}}_{Ro}$ results changing the updated system matrices by pre-multiply by some non-singular matrix \mathbf{X} .

$$[\mathbf{K}_o \quad \mathbf{D}_o \quad \mathbf{M}_o] = \mathbf{X} [\mathbf{K}_{up} \quad \mathbf{D}_{up} \quad \mathbf{M}_{up}] \quad (8.101)$$

The matrix \mathbf{X} can be calculated from equation (8.101).

$$\mathbf{X} = ([\mathbf{K}_o \quad \mathbf{D}_o \quad \mathbf{M}_o]) ([\mathbf{K}_{up} \quad \mathbf{D}_{up} \quad \mathbf{M}_{up}])^+ \quad (8.102)$$

The superscript “+” represents the pseudo inverse of the updated numerical model matrices. The improved numerical models $\{\mathbf{K}_{imp}, \mathbf{D}_{imp}, \mathbf{M}_{imp}\}$ can be found using equation (8.102).

$$\begin{aligned} \mathbf{K}_{imp} &= \mathbf{X} \mathbf{K}_{up} \\ \mathbf{D}_{imp} &= \mathbf{X} \mathbf{D}_{up} \\ \mathbf{M}_{imp} &= \mathbf{X} \mathbf{M}_{up} \end{aligned} \quad (8.103)$$

8.3.4 Model updating based on cross orthogonality measures for general damped systems

The different cross orthogonality measures $\{\underline{\mathbf{XO}}_{Ma}, \underline{\mathbf{XO}}_{Da}, \underline{\mathbf{XO}}_{Ka}\}$ are affected when the system matrices $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$ are changed. Extending the ideas for undamped systems in section 8.2.2 to general damped systems, the different cross orthogonality measures $\{\underline{\mathbf{XO}}_{Ma}, \underline{\mathbf{XO}}_{Da}, \underline{\mathbf{XO}}_{Ka}\}$ which are introduced in Chapter 7, section 7.5.2 can be expressed as

$$\underline{\mathbf{XO}}_{Ma} = \underline{\mathbf{XO}}_M + \Delta \underline{\mathbf{XO}}_M = \underline{\mathbf{XO}}_M + \sum_{j=1}^p \frac{\partial \underline{\mathbf{XO}}_{Ma}}{\partial \theta_j} \Delta \theta_j \quad (8.104)$$

$$\underline{\mathbf{XO}}_{Da} = \underline{\mathbf{XO}}_D + \Delta \underline{\mathbf{XO}}_D = \underline{\mathbf{XO}}_D + \sum_{j=1}^p \frac{\partial \underline{\mathbf{XO}}_{Da}}{\partial \theta_j} \Delta \theta_j \quad (8.105)$$

$$\underline{\mathbf{XO}}_{Ka} = \underline{\mathbf{XO}}_K + \Delta \underline{\mathbf{XO}}_K = \underline{\mathbf{XO}}_K + \sum_{j=1}^p \frac{\partial \underline{\mathbf{XO}}_{Ka}}{\partial \theta_j} \Delta \theta_j \quad (8.106)$$

Equations (7.103), (7.107) and (7.111) in Chapter 7 section 7.5.3 require modal information from both systems $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ and $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$. The modes of the calculated system must be correctly matched to the modes of the true system. The introduced procedures in section 8.3.2.1 are used for scaling and ordering the analytical diagonalising SPEs.

8.3.4.1 Residual measures based on cross orthogonality

The residuals $(\underline{\mathbf{XO}}_{Ma} - \underline{\mathbf{J}})$, $(\underline{\mathbf{XO}}_{Da} - \underline{\mathbf{J}})$ and $(\underline{\mathbf{XO}}_{Ka} - \underline{\mathbf{J}})$ are measures of the discrepancies between systems $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ and $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$. The vector of residuals \mathbf{r} has dimension $(12n^2)$, and it can be written as

$$\mathbf{r} = \begin{bmatrix} \text{vec}(\underline{\mathbf{XO}}_{Ma} - \underline{\mathbf{J}}) \\ \text{vec}(\underline{\mathbf{XO}}_{Da} - \underline{\mathbf{J}}) \\ \text{vec}(\underline{\mathbf{XO}}_{Ka} - \underline{\mathbf{J}}) \end{bmatrix} \quad (8.107)$$

In the updating method proposed, we require to determine the sensitivity matrix, \mathbf{S} , which relates small changes in the parameters to corresponding small changes in the vector of residuals \mathbf{r} . The sensitivity could be computed numerically. In practice, we would normally compute it analytically. The rates of change of different *cross*

orthogonality measures are introduced in Chapter 7 section 7.5.4.1 in equations (7.124), (7.125) and (7.126).

8.3.5 *Model updating based on mutual orthogonality measures*

The *mutual orthogonality* measures are introduced in Chapter 7, section 7.5.3 in equations (7.105), (7.109) and (7.113). These measures are independent of any modal information from the system $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$. However, for systems having non-symmetric matrices, both right and left modes for the measured system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ are required.

8.3.5.1 *Residual measures based on mutual orthogonality measures for general damped systems*

The residuals $(\underline{\mathbf{O}}_{\underline{Moo}} - \underline{\mathbf{J}})$, $(\underline{\mathbf{O}}_{\underline{Doo}} - \underline{\mathbf{J}})$ and $(\underline{\mathbf{O}}_{\underline{Koo}} - \underline{\mathbf{J}})$ are measures of the differences between systems $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ and $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$. If these measures are zeros, then the matrices $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ and $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$ are identical, and $\underline{\mathbf{O}}_{\underline{Moo}} = \underline{\mathbf{J}}$, $\underline{\mathbf{O}}_{\underline{Doo}} = \underline{\mathbf{J}}$ and $\underline{\mathbf{O}}_{\underline{Koo}} = \underline{\mathbf{J}}$.

Equations (7.105), (7.109) and (7.113) require invertibility of the matrices $\{\underline{\mathbf{M}}_{\underline{Do}}, \underline{\mathbf{D}}_{\underline{Do}}, \underline{\mathbf{K}}_{\underline{Do}}\}$. Methods are investigated for model updating. These methods which are applied for undamped systems are introduced in Chapter 7 section 7.3.5. The general ideas for residual measures of undamped systems in section 8.2.3 are extended to generally damped systems. The residuals measures do not demand

invertibility of $\{\underline{\mathbf{M}}_{Do}, \underline{\mathbf{D}}_{Do}, \underline{\mathbf{K}}_{Do}\}$ (further details in Chapter 7, section 7.5.5.2). The vector of residuals can be written as

$$\mathbf{r} = \begin{bmatrix} \text{vec}\left(\underline{\mathbf{O}}_M - \begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A} & \mathbf{B} \end{bmatrix}\right) \\ \text{vec}\left(\underline{\mathbf{O}}_D - \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & -\mathbf{A} \end{bmatrix}\right) \\ \text{vec}\left(\underline{\mathbf{O}}_K - \begin{bmatrix} -\mathbf{B} & -\mathbf{A} \\ -\mathbf{A} & \mathbf{0} \end{bmatrix}\right) \end{bmatrix} \quad (8.108)$$

Matrices $\{\underline{\mathbf{O}}_M, \underline{\mathbf{O}}_D, \underline{\mathbf{O}}_K\}$ and $\{\mathbf{A}, \mathbf{B}\}$ are well developed in Chapter 7 section 7.5.5.1.

The matrices $\{\mathbf{A}, \mathbf{B}\}$ represent palindromic polynomials for the second systems $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$.

$$\mathbf{A} = \Psi_{Lo} \Psi_{Ro} \Theta_{Lo} \Theta_{Ro} = \mathbf{K}_{Do} \mathbf{M}_{Do} \quad (8.109)$$

$$\mathbf{B} = \Pi_o (\Psi_{Lo} \Theta_{Ro} + \Psi_{Ro} \Theta_{Lo}) (\Psi_{Lo} \Psi_{Ro} + \Theta_{Lo} \Theta_{Ro}) = \mathbf{D}_{Do} (\mathbf{K}_{Do} + \mathbf{M}_{Do}) \quad (8.110)$$

The sensitivity matrix ‘S’ can be determined either numerically or analytically. It is possible to compute the sensitivity analytically. The rate of change of *mutual orthogonality* measures are introduced in Chapter 7, section 7.5.5 in equations (7.148), (149) and (150). These derivatives are independent of any information from system $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$ and do not require invertibility of $\{\underline{\mathbf{M}}_{Do}, \underline{\mathbf{D}}_{Do}, \underline{\mathbf{K}}_{Do}\}$.

8.4 Example-1

This example considers a six degree of freedom system $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$. Figure 8-2 shows a physical system structure that has 6 lumped masses $\{m_1, m_2, m_3, m_4, m_5, m_6\}$ which are linked by 14 springs $\{k_{01}, k_{12}, k_{13}, k_{16}, k_{02}, k_{23}, k_{24}, k_{03}, k_{34}, k_{35}, k_{45}, k_{46}, k_{56}, k_{06}\}$ and 14 dampers $\{d_{01}, d_{12}, d_{13}, d_{16}, d_{02}, d_{23}, d_{24}, d_{03}, d_{34}, d_{35}, d_{45}, d_{46}, d_{56}, d_{06}\}$.

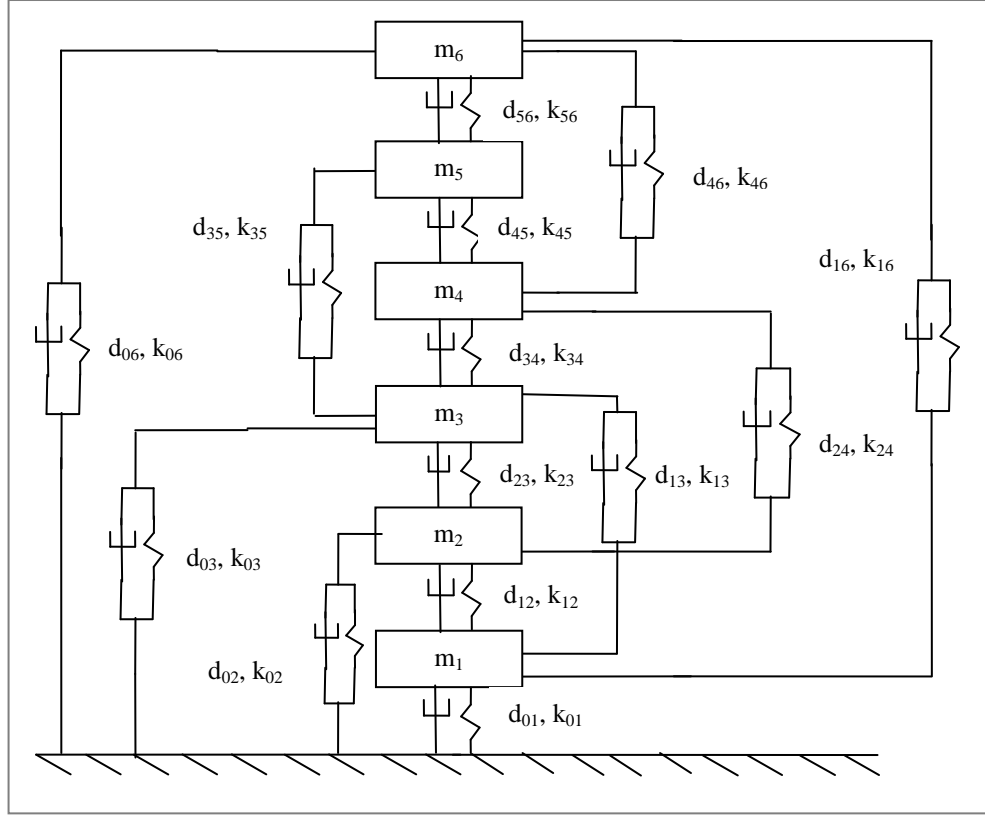


Figure 8-2 : Physical system structure

This example is deliberately chosen for model updating based on homogenous coordinates when only one side (the right side) of the measured modes are known. The total number of unknowns is 34 and there are 144 residual entries ($2n \times 2n$) whereas 108 of these are independent ($3n^2$). The mass matrix is diagonal, the stiffness and damping matrices are symmetric. The system are shown below

$$\mathbf{K}_o = \begin{bmatrix} K_{11} & -k_{12} & -k_{13} & 0 & 0 & -k_{16} \\ -k_{12} & K_{22} & -k_{23} & -k_{24} & 0 & 0 \\ -k_{13} & -k_{23} & K_{33} & -k_{34} & -k_{35} & 0 \\ 0 & -k_{24} & -k_{34} & K_{44} & -k_{45} & -k_{46} \\ 0 & 0 & -k_{35} & -k_{45} & K_{55} & -k_{56} \\ -k_{16} & 0 & 0 & -k_{46} & -k_{56} & K_{66} \end{bmatrix} \quad (8.111)$$

where the diagonal entries $\{K_{11}, K_{22}, K_{33}, K_{44}, K_{55}, K_{66}\}$ of the stiffness matrix \mathbf{K}_o are

$$\begin{aligned}
K_{11} &= k_{01} + k_{12} + k_{13} + k_{16} \\
K_{22} &= k_{02} + k_{12} + k_{23} + k_{24} \\
K_{33} &= k_{03} + k_{13} + k_{23} + k_{34} + k_{35} \\
K_{44} &= k_{24} + k_{34} + k_{45} + k_{46} \\
K_{55} &= k_{35} + k_{45} + k_{56} \\
K_{66} &= k_{16} + k_{46} + k_{56} + k_{06}
\end{aligned}$$

$$\mathbf{D}_o = \begin{bmatrix} D_{11} & -d_{12} & -d_{13} & 0 & 0 & -d_{16} \\ -d_{12} & D_{22} & -d_{23} & -d_{24} & 0 & 0 \\ -d_{13} & -d_{23} & D_{33} & -d_{34} & -d_{35} & 0 \\ 0 & -d_{24} & -d_{34} & D_{44} & -d_{45} & -d_{46} \\ 0 & 0 & -d_{35} & -d_{45} & D_{55} & -d_{56} \\ -d_{16} & 0 & 0 & -d_{46} & -d_{56} & D_{66} \end{bmatrix} \quad (8.112)$$

where the diagonal entries $\{D_{11}, D_{22}, D_{33}, D_{44}, D_{55}, D_{66}\}$ of the stiffness matrix \mathbf{D}_o are

$$\begin{aligned}
D_{11} &= d_{01} + d_{12} + d_{13} + d_{16} \\
D_{22} &= d_{02} + d_{12} + d_{23} + d_{24} \\
D_{33} &= d_{03} + d_{13} + d_{23} + d_{34} + d_{35} \\
D_{44} &= d_{24} + d_{34} + d_{45} + d_{46} \\
D_{55} &= d_{35} + d_{45} + d_{56} \\
D_{66} &= d_{16} + d_{46} + d_{56} + d_{06}
\end{aligned}$$

$$\mathbf{M}_o = \text{diag}(m_1, m_2, m_3, m_4, m_5, m_6) \quad (8.113)$$

The parameters are generated randomly as follows

$$\begin{aligned}
&\left\{ \begin{aligned} k_{01} &= 405, k_{12} = 399, k_{13} = 123, k_{16} = 221, k_{02} = 129, k_{23} = 330, k_{24} = 490, \\ k_{03} &= 132, k_{34} = 458, k_{35} = 41, k_{45} = 65, k_{46} = 267, k_{56} = 376, k_{06} = 483 \end{aligned} \right\} \quad \text{N/m.} \\
&\left\{ \begin{aligned} d_{01} &= 34, d_{12} = 6, d_{13} = 23, d_{16} = 16, d_{02} = 1, d_{23} = 35, d_{24} = 47, \\ d_{03} &= 45, d_{34} = 14, d_{35} = 4, d_{45} = 12, d_{46} = 18, d_{56} = 42, d_{06} = 7 \end{aligned} \right\} \quad \text{Ns/m.} \\
&\{m_1 = 89, m_2 = 22, m_3 = 52, m_4 = 89, m_5 = 66, m_6 = 13\} \quad \text{kg.}
\end{aligned}$$

These parameters can be arranged in a vector θ_0 .

All of the eigenvalues and the eigenvectors form complex conjugate pairs.

$$\lambda_o = \begin{bmatrix} -0.1307 \pm 1.6375i \\ -0.4049 \pm 2.6123i \\ -0.5242 \pm 3.5790i \\ -1.0827 \pm 4.8472i \\ -2.2592 \pm 7.7762i \\ -3.3712 \pm 9.7264i \end{bmatrix}$$

The right and left diagonalising SPEs $\{\underline{\mathbf{T}}_{Ro}, \underline{\mathbf{T}}_{Lo}\}$ are identical $\underline{\mathbf{T}}_{Ro} = \underline{\mathbf{T}}_{Lo}$. The diagonalised system matrices $\{\mathbf{K}_{Do}, \mathbf{D}_{Do}, \mathbf{M}_{Do}\}$ are

\mathbf{K}_{Do}	\mathbf{D}_{Do}	\mathbf{M}_{Do}
0.9979	0.0635	0.0094
0.9992	0.0687	0.0152
0.9975	0.0874	0.0404
0.9939	0.0796	0.0759
0.9338	0.0905	0.3461
0.9835	0.11397	0.1407

The analytical system matrices $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$ are changed by $\{\Delta\mathbf{K}, \Delta\mathbf{D}, \Delta\mathbf{M}\}$. In this example, σ is scalar value and selected equal 1.0e-2.

$$\mathbf{K}_a = \mathbf{K}_o + \sigma\mathbf{K}_1$$

$$\mathbf{D}_a = \mathbf{D}_o + \sigma\mathbf{D}_1$$

$$\mathbf{M}_a = \mathbf{M}_o + \sigma\mathbf{M}_1$$

Matrices $\{\mathbf{K}_1, \mathbf{D}_1, \mathbf{M}_1\}$ are generated randomly with the same structure as the system matrices $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ as follows. In other words, the matrices \mathbf{K}_a , \mathbf{D}_a and \mathbf{M}_a are consistent with equations (8.111), (8.112) and (8.113) respectively.

$$\mathbf{K}_1 = \begin{bmatrix} 2504 & -756 & -749 & 0 & 0 & -568 \\ -756 & 1790 & -209 & -718 & 0 & 0 \\ -749 & -209 & 3588 & -677 & -955 & 0 \\ 0 & -718 & -677 & 2406 & -773 & -822 \\ 0 & 0 & -955 & -773 & 2550 & -822 \\ -568 & 0 & 0 & -238 & -822 & 2597 \end{bmatrix}, \quad \mathbf{D}_1 = \begin{bmatrix} 241 & -63 & -21 & 0 & 0 & -72 \\ -63 & 215 & -39 & -59 & 0 & 0 \\ -21 & -39 & 257 & -41 & -86 & 0 \\ 0 & -59 & -41 & 206 & -45 & -61 \\ 0 & 0 & -86 & -45 & 188 & -57 \\ -72 & 0 & 0 & -61 & -57 & 227 \end{bmatrix}$$

$$\mathbf{M}_1 = \begin{bmatrix} 58 & 0 & 0 & 0 & 0 & 0 \\ 0 & 169 & 0 & 0 & 0 & 0 \\ 0 & 0 & 24 & 0 & 0 & 0 \\ 0 & 0 & 0 & 65 & 0 & 0 \\ 0 & 0 & 0 & 0 & 123 & 0 \\ 0 & 0 & 0 & 0 & 0 & 153 \end{bmatrix}$$

In this example, there are 34 unknowns in the parameter vector $\Delta\theta$ which we hope to find. The initial values of variable parameters are zero.

The right diagonalising SPE $\underline{\mathbf{T}}_{Ro}$ for system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ is used to calculate residuals vector. The residuals are determined based on a linear combination of *LAMs* according to equation (8.99). We noted that the sensitivity matrix is rank deficient (rank is less than number of unknowns). This leads to under-determined set of equations. The sensitivity matrix has dimensions (144×34). Each column \mathbf{s}_{ci} where $i=1:34$ in the sensitivity matrix has entries according to changing each entry in the parameter vector $\Delta\theta$.

Applying the process described in section 8.3.3. The sum of squares of residuals approaches zero and equation (8.98) is satisfied. The number of iterations to convergence depends on how far the initial values of the variable parameters are from the exact values. However, the updated system matrices $\{\mathbf{K}_{up}, \mathbf{D}_{up}, \mathbf{M}_{up}\}$ do not necessary match the measured system matrices $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$. The matrices $\{\mathbf{K}_{up}, \mathbf{D}_{up}, \mathbf{M}_{up}\}$ shown below are rounded.

$$\mathbf{K}_{up} = \begin{bmatrix} 1168 & -406 & -125 & 0 & 0 & -225 \\ -406 & 1371 & -336 & -498 & 0 & 0 \\ -125 & -336 & 1103 & -466 & -42 & 0 \\ 0 & -498 & -466 & 1302 & -66 & -272 \\ 0 & 0 & -42 & -66 & 490 & -382 \\ -225 & 0 & 0 & -272 & -382 & 1370 \end{bmatrix}$$

$$\mathbf{D}_{up} = \begin{bmatrix} 80 & -6 & -23 & 0 & 0 & -16 \\ -6 & 91 & -36 & -48 & 0 & 0 \\ -23 & -36 & 123 & -14 & -4 & 0 \\ 0 & -48 & -14 & 93 & -12 & -18 \\ 0 & 0 & -4 & -12 & 59 & -43 \\ -16 & 0 & 0 & -18 & -43 & 84 \end{bmatrix}, \quad \mathbf{M}_{up} = \begin{bmatrix} 91 & 0 & 0 & 0 & 0 & 0 \\ 0 & 22 & 0 & 0 & 0 & 0 \\ 0 & 0 & 53 & 0 & 0 & 0 \\ 0 & 0 & 0 & 91 & 0 & 0 \\ 0 & 0 & 0 & 0 & 67 & 0 \\ 0 & 0 & 0 & 0 & 0 & 13 \end{bmatrix}$$

Comparing equation (8.98) for updated models with equation (8.88) for the measured system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ for the measured right modes $\underline{\mathbf{T}}_{Ro}$ results changing the updated system matrices $\{\mathbf{K}_{up}, \mathbf{D}_{up}, \mathbf{M}_{up}\}$ by pre-multiply by some non-singular matrix \mathbf{X} as in equation (8.101). The matrix \mathbf{X} is calculated from equation (8.102). This matrix is diagonal and all entries are identical.

$$\mathbf{X} = \text{diag} \begin{pmatrix} 0.98311 \\ 0.98311 \\ 0.98311 \\ 0.98311 \\ 0.98311 \\ 0.98311 \end{pmatrix}$$

The improved analytical models are calculated using equation (8.103)

$$\mathbf{K}_{imp} = \mathbf{X}\mathbf{K}_{up}, \quad \mathbf{D}_{imp} = \mathbf{X}\mathbf{D}_{up}, \quad \mathbf{M}_{imp} = \mathbf{X}\mathbf{M}_{up}$$

8.5 Example-2

In this example, we consider a system with general linear damping having four degrees of freedom. The system has been deliberately chosen to have non symmetric matrices $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$. The aim of this example is to update the analytical system

matrices $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$ by driving the vector of residuals \mathbf{r} to zero. For updating parameters, the information required is the diagonalised system $\{\mathbf{K}_{Do}, \mathbf{D}_{Do}, \mathbf{M}_{Do}\}$, the diagonalising transformations $\{\underline{\mathbf{T}}_{Lo}, \underline{\mathbf{T}}_{Ro}\}$ for the measured system and the system matrices for the analytical system $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$. The left and right diagonalising transformations $\{\underline{\mathbf{T}}_{Lo}, \underline{\mathbf{T}}_{Ro}\}$ for the measured system simultaneously diagonalise $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$. We consider 6 unknown parameters $\{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6\}$ and we suppose that the diagonalised system matrices $\{\mathbf{K}_{Do}, \mathbf{M}_{Do}\}$ are invertible.

$$\mathbf{K}_o = \begin{bmatrix} -489 & -463 & 396 & 295 \\ 471 & 393 & -81 & -361 \\ -282 & -373 & -422 & 212 \\ -494 & -479 & -21 & 246 \end{bmatrix}, \quad \mathbf{D}_o = \begin{bmatrix} 10 & 11 & 24 & -21 \\ 7 & -4 & -18 & 18 \\ -14 & 5 & -7 & -18 \\ -3 & -12 & 11 & -12 \end{bmatrix}, \quad \mathbf{M}_o = \begin{bmatrix} -46 & 30 & 38 & -25 \\ -39 & -43 & 2 & 2 \\ 4 & -41 & 27 & -19 \\ -3 & 4 & -7 & 25 \end{bmatrix}$$

The diagonalised system $\{k_{io}, d_{io}, m_{io}\}$ is summarised below.

k_{io}	d_{io}	m_{io}
0.6627	0.6858	0.3009
0.6112	-0.7534	0.2426
0.9392	-0.3375	0.0638
0.9335	0.3531	0.0622

Scaling has been applied to make $k_{io}^2 + d_{io}^2 + m_{io}^2 = 1$ for $1 \leq i \leq 4$.

The entries of the diagonalised system matrices $\{\mathbf{K}_{Do}, \mathbf{D}_{Do}, \mathbf{M}_{Do}\}$ for the original $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ can be factorised as equations (7.79)-(7.81) indicate. Matrices $\{\underline{\Theta}_{Lo}, \underline{\Psi}_{Lo}\}$, right $\{\underline{\Theta}_{Ro}, \underline{\Psi}_{Ro}\}$ and Π_o satisfy equations (7.82)-(7.84).

$$\underline{\Theta}_{Lo} = \underline{\Theta}_{Ro} = \text{diag} \left(\begin{bmatrix} 0.5486 \\ 0.4925 \\ 0.2525 \\ 0.2494 \end{bmatrix} \right), \quad \underline{\Psi}_{Lo} = \underline{\Psi}_{Ro} = \text{diag} \left(\begin{bmatrix} 0.8141 \\ 0.7818 \\ 0.9691 \\ 0.9662 \end{bmatrix} \right), \quad \Pi_o = \text{diag} \left(\begin{bmatrix} 0.7678 \\ -0.9783 \\ -0.6895 \\ 0.7326 \end{bmatrix} \right)$$

The analytical system matrices $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$ are related to the true system matrices by

$$\mathbf{K}_a = \mathbf{K}_o + \mathbf{K}_\theta = \mathbf{K}_o + \theta_1 \mathbf{v}\mathbf{v}^T + \theta_2 \mathbf{w}\mathbf{w}^T$$

$$\mathbf{D}_a = \mathbf{D}_o + \mathbf{D}_\theta = \mathbf{D}_o + \theta_3 \mathbf{v}\mathbf{v}^T + \theta_4 \mathbf{w}\mathbf{w}^T$$

$$\mathbf{M}_a = \mathbf{M}_o + \mathbf{M}_\theta = \mathbf{M}_o + \theta_5 \mathbf{v}\mathbf{v}^T + \theta_6 \mathbf{w}\mathbf{w}^T$$

where $\mathbf{v} = [1 \ 0 \ 1 \ 0]^T$, $\mathbf{w} = [1 \ 0 \ -1 \ 1]^T$ and where $\{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6\}$ are scalar parameters.

We can attempt to update the system matrices $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$ from any starting point.

We consider that the scalar parameters $\{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6\}$ are six points on a hypersphere having constant radius. To test the updating, we chose many different random vectors $\boldsymbol{\theta}$ and scaled each one such that

$$\boldsymbol{\theta}^T \boldsymbol{\theta} = r^2 \quad (8.114)$$

and

$$\boldsymbol{\theta} = [\theta_1 \ \theta_2 \ \theta_3 \ \theta_4 \ \theta_5 \ \theta_6]^T \quad (8.115)$$

For any updating method, we define the *ball of convergence* as the set of all vectors $\boldsymbol{\theta}$ for which the process will converge to $\boldsymbol{\theta} = 0$. The radius of this ball of convergence is defined as the minimum value of $(\boldsymbol{\theta}^T \boldsymbol{\theta})^{1/2}$ outside this set.

For comparison purposes, we investigated updating the system based on the eigenvectors of the analytical system $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$ and based on the mutual orthogonality measures described in section 8.3.5. The updating procedure based on eigenvectors described in section 8.3.1 has been tested. It converges however; the

radius of the ball of convergence is small. The updating procedure based on *mutual orthogonality* has been tested many times. It always converges monotonically for all starting values of θ and the radius of the ball of convergence is thus infinity. Convergence is in this example always achieved in 6 iteration or less.

Figure 8-3 shows a comparison of the distance between the true system matrices $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ and the analytical system matrices $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$. The radii r_{mut} and r_{eig} show the maximum ball of convergence for model updating based on mutual orthogonality and eigenvectors, respectively. The maximum range of r_{eig} is 9.641 whereas the maximum range of r_{mut} goes to infinity $r_{mut} \rightarrow \infty$. The range of radius r_{eig} lies inside the range of radius r_{mut} .

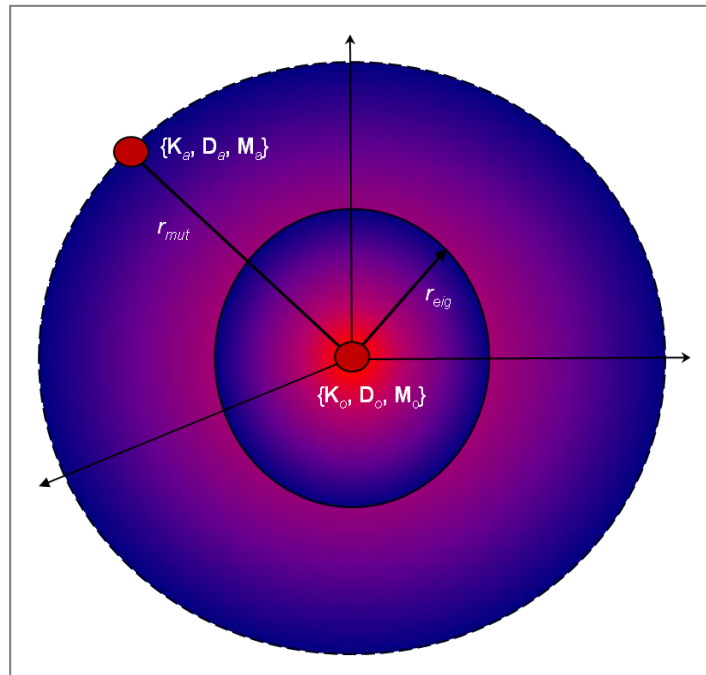


Figure 8-3 : Radius of ball of convergence

8.6 Conclusion

This chapter has developed methods which useful for the model updating of general damped linear systems. Model updating procedures based on eigenvectors, diagonalising SPEs and both *cross-orthogonality* measures and mutual-orthogonality measures are presented. In the cases of both *cross-orthogonality* and *mutual-orthogonality* measures, these measures are defined in terms of real matrices - the diagonalising structure preserving equivalences DSPEs.

If *cross-orthogonality* measures are to be used, then (in effect) the modes of the calculated system must be correctly matched to the modes of the true system. By contrast, if *mutual orthogonality* measures are employed, this matching of modes is unnecessary. However, for systems having non-symmetric matrices, both left and right modes are required for the *mutual orthogonality* measures.

Example-1 presented model updating based on homogenous coordinates. The linear combination of cross orthogonality checks are used to form the residuals for updating model parameters. The model updating procedures require right or left modal information for the physical system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ and analytical system matrices $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$. An improved step is required to match the updated model matrices with the measured system because the sensitivity matrix is rank deficient.

In Example-2, model updating based on the mutual orthogonality measures exhibits monotonic convergence from every starting position. That is to say, the ball of

convergence has an infinite radius. This contrasts starkly with updating procedures based on comparing eigenvectors which exhibit a finite ball of convergence.

The following chapter develops Craig Bampton transformations as model reduction method.

CHAPTER 9. Craig-Bampton Transformations (CBTs)

This chapter develops Craig Bampton transformations for undamped system. The ideas are extended for generally damped systems in state space with arbitrary input and output matrices and without the assumption of symmetry. Dynamic behaviours of large complex structures are typically modelled using finite element analysis. In order to analyse the response of any structure, the structure is divided into a large discrete number of substructures or components. The substructures or components modes are obtained. The substructures models are coupled to give reduced dimension system models [78].

Dynamic substructuring is mainly used to couple reduced models of complex structures, in test verification of finite element models of components, or to implement computation of the dynamics of very large finite element models [1, 3]. Several literatures are published on substructure coupling methods [78, 79, 143, 144]. Some methods are developed for undamped and damped systems and used for large complex structural systems [79].

Craig and Bampton [80] proposed a technique for representing substructures. Craig-Bampton transformations (CBTs) are divided into two parts [80]. The first part of CBTs represents boundary generalized coordinates which are related to the displacement modes of the substructures known as *constraint modes*. The second part of CBTs represents substructure *normal mode* generalized coordinates which

describe the motion of the interior coordinates relative to the fixed boundaries of the substructures.

9.1 Undamped substructure systems

The equations of motion for any undamped symmetric substructure can be written as

$$\begin{aligned}\mathbf{f}_s &= \mathbf{S}_s \mathbf{u}_s \\ \mathbf{M}_s \ddot{\mathbf{q}}_s(t) + \mathbf{K}_s \mathbf{q}_s(t) &= \mathbf{f}_s(t) \\ \mathbf{y}_s &= \mathbf{S}_s^T \mathbf{q}_s\end{aligned}\tag{9.1}$$

where \mathbf{S}_s represents selection matrix at terminal degrees of freedom (DoFs) of the substructure. Subscript “s” denotes substructure or component. $\mathbf{M}_s = \mathbf{M}_s^T$, $\mathbf{K}_s = \mathbf{K}_s^T$ and \mathbf{q}_s are the substructure’s mass, stiffness matrices, and displacement vector respectively. Vector \mathbf{f}_s includes external applied forces and the forces that act on the substructure due to its connection to adjacent components at boundary DoFs. The physical displacement coordinates \mathbf{q}_s are represented in terms of substructure generalized coordinates \mathbf{q}_{sn} by the coordinate transformation

$$\mathbf{q}_s = \mathbf{T} \mathbf{q}_{sn}, \quad \mathbf{f}_{sn} = \mathbf{T}^T \mathbf{f}_s\tag{9.2}$$

The matrix \mathbf{T} is a coordinate transformation populated with preselected substructure modes, (including rigid body modes, normal modes of free vibration, constraint modes, and attachment modes). The substructure equation of motion in generalized coordinates is defined as

$$\begin{aligned}\mathbf{f}_{sn} &= \mathbf{S}_{sn} \mathbf{u}_s \\ \mathbf{M}_{sn} \ddot{\mathbf{q}}_{sn}(t) + \mathbf{K}_{sn} \mathbf{q}_{sn}(t) &= \mathbf{f}_{sn}(t) \\ \mathbf{y}_s &= \mathbf{S}_{sn}^T \mathbf{q}_{sn}\end{aligned}\tag{9.3}$$

where $\mathbf{S}_{sn} = \mathbf{T}^T \mathbf{S}_s$, and

$$\mathbf{M}_{sn} = \mathbf{T}^T \mathbf{M}_s \mathbf{T}\tag{9.4}$$

$$\mathbf{K}_{sn} = \mathbf{T}^T \mathbf{K}_s \mathbf{T} \quad (9.5)$$

The form of the equation of motion for a substructure given in equation (9.1) is partitioned into boundary DoFs and interior DoFs which can be expressed as

$$\begin{aligned} \begin{Bmatrix} \mathbf{f}_b \\ \mathbf{f}_i \end{Bmatrix} &= \begin{bmatrix} \mathbf{S}_b \\ \mathbf{0} \end{bmatrix} \{\mathbf{u}_s\} \\ \begin{bmatrix} \mathbf{M}_{bb} & \mathbf{M}_{bi} \\ \mathbf{M}_{ib} & \mathbf{M}_{ii} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}}_b \\ \ddot{\mathbf{q}}_i \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{bb} & \mathbf{K}_{bi} \\ \mathbf{K}_{ib} & \mathbf{K}_{ii} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_b \\ \mathbf{q}_i \end{Bmatrix} &= \begin{Bmatrix} \mathbf{f}_b \\ \mathbf{f}_i \end{Bmatrix} \\ \{\mathbf{y}_s\} &= \begin{bmatrix} \mathbf{S}_b \\ \mathbf{0} \end{bmatrix}^T \begin{Bmatrix} \mathbf{q}_b \\ \mathbf{q}_i \end{Bmatrix} \end{aligned} \quad (9.6)$$

where subscripts b and i represent the boundary and interior DoFs, \mathbf{q}_b represents a vector of boundary DoFs, \mathbf{q}_i represents internal DoFs. Figure 9-1 shows external or boundary DoFs and the internal DoFs of an arbitrary substructure or component.

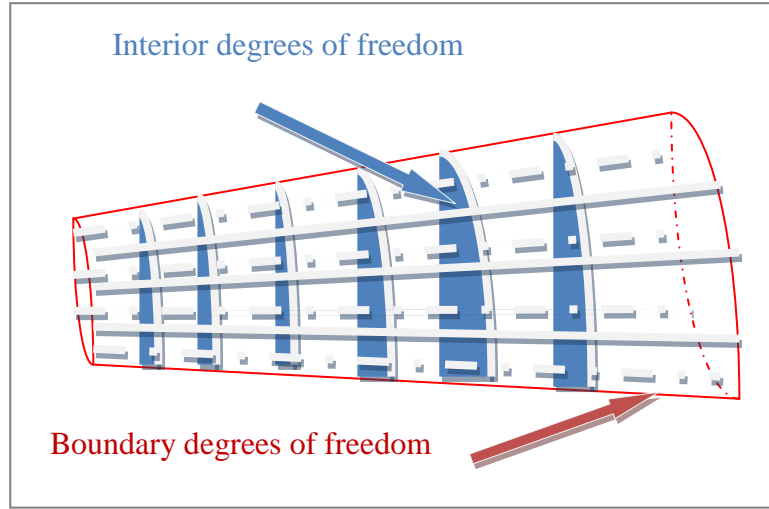


Figure 9-1 : Substructure boundary and interior DoFs

9.2 Craig-Bampton transformations for undamped substructure systems

Craig-Bampton transformations for undamped systems are introduced in Chapter 2 section 2.7.7. These transformations are used as a model reduction method. The

Craig-Bampton transformations, like any other coordinate transformation are defined as

$$\mathbf{q}_s = \mathbf{T}_{Rcb} \mathbf{q}_{sn} \quad (9.7)$$

$$\mathbf{f}_{sn} = \mathbf{T}_{Lcb}^T \mathbf{f}_s \quad (9.8)$$

In fact, Craig-Bampton transformations are more specific since $\mathbf{T}_{Rcb} = \mathbf{T}_{Lcb} = \mathbf{T}_{cb}$. Now, in Craig-Bampton, the transformation matrix, \mathbf{T}_{cb} comprises two separate parts

$$\mathbf{T}_{cb} = [\mathbf{F} \quad \mathbf{E}] \quad (9.9)$$

The \mathbf{F} part ensures that the substructure model will reproduce static behaviour exactly. The \mathbf{E} part ensures that if the substructure model has all of its terminal DoFs clamped, it will reproduce at least the first few natural modes correctly. Before discussing these two parts separately it is necessary to discuss matrices which span the space orthogonal to \mathbf{S}_s .

9.2.1 Matrix spanning the space orthogonal to \mathbf{S}_s for undamped substructure systems

In the usual presentation of Craig-Bampton, it is implicit that \mathbf{S}_s is defined as shown below. \mathbf{S}_s has dimensions $(n \times n_b)$ where n_b is number of DoFs at the terminals DoFs.

$$\mathbf{S}_s = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \hline 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \quad (9.10)$$

Consider that some $(n \times (n - n_b))$ matrix, \mathbf{R}_s , can be found such that : $[\mathbf{S}_s \quad \mathbf{R}_s]$ is a square invertible matrix and such that

$$\mathbf{S}_s^T \mathbf{R}_s = \mathbf{0} \quad (9.11)$$

The matrix \mathbf{R}_s will be used later.

9.2.2 *Finding the matrix \mathbf{F} for undamped substructure systems*

The first part of the Craig-Bampton transformation \mathbf{F} defines the constraint modes Φ^c . These modes are defined as the static deformation of a substructure when a unit displacement is applied to the boundary coordinates and the interior degrees of freedom are not forced

$$\mathbf{F} = \begin{bmatrix} \mathbf{I}_{bb} \\ \Phi^c \end{bmatrix} \quad (9.12)$$

Note that the dimensions of \mathbf{F} are fixed. In words, \mathbf{F} represents a mapping between a set of prescribed displacements at the terminals of a substructure and the complete set of deflections of the substructure which results in minimum potential energy. \mathbf{F} has the same number of columns as \mathbf{S}_s and it can be derived as follows:

Let \mathbf{a} represent some arbitrary vector describing the prescribed deflections at the terminals, \mathbf{a} has dimensions $(n_b \times 1)$. From equation (9.1) we get

$$\mathbf{a} = \mathbf{S}_s^T \mathbf{q}_s \quad (9.13)$$

The space of all vectors, \mathbf{q}_s , which can satisfy equation (9.13) is given by

$$\mathbf{q}_s = \mathbf{S}_s \left(\mathbf{S}_s^T \mathbf{S}_s \right)^{-1} \mathbf{a} + \mathbf{R}_s \mathbf{b} =: \mathbf{Q}_s \mathbf{a} + \mathbf{R}_s \mathbf{b} \quad (9.14)$$

where \mathbf{b} is some arbitrary vector with dimensions $((n-n_b) \times 1)$. Now, we can easily see that (twice) the potential energy is given by

$$\begin{aligned} 2\mathfrak{P}\mathfrak{E} &= \mathbf{q}_s^T \mathbf{K}_s \mathbf{q}_s = \begin{bmatrix} \mathbf{a}^T & \mathbf{b}^T \end{bmatrix} \begin{bmatrix} \mathbf{Q}_s^T \\ \mathbf{R}_s^T \end{bmatrix} \mathbf{K}_s \begin{bmatrix} \mathbf{Q}_s & \mathbf{R}_s \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{a}^T & \mathbf{b}^T \end{bmatrix} \begin{bmatrix} \mathbf{K}_{bb} & \mathbf{K}_{bi} \\ \mathbf{K}_{ib} & \mathbf{K}_{ii} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \end{aligned} \quad (9.15)$$

$$\begin{aligned} \mathbf{K}_{bb} &= \mathbf{Q}_s^T \mathbf{K}_s \mathbf{Q}_s, & \mathbf{K}_{bi} &= \mathbf{Q}_s^T \mathbf{K}_s \mathbf{R}_s \\ \mathbf{K}_{ib} &= \mathbf{R}_s^T \mathbf{K}_s \mathbf{Q}_s, & \mathbf{K}_{ii} &= \mathbf{R}_s^T \mathbf{K}_s \mathbf{R}_s \end{aligned} \quad (9.16)$$

Now, following Guyan's logic [52], it is easy to see that for any given vector, \mathbf{a} , the potential energy is minimised $\frac{\partial(2\mathfrak{P}\mathfrak{E})}{\partial \mathbf{b}} = 0$ by choosing:

$$\mathbf{b} = -(\mathbf{K}_{ii}^{-1} \mathbf{K}_{ib}) \mathbf{a} \quad (9.17)$$

Substitute all of this back into equation (9.14) to find

$$\begin{aligned} \mathbf{q}_s &= \left(\mathbf{Q}_s - \mathbf{R}_s (\mathbf{K}_{ii}^{-1} \mathbf{K}_{ib}) \right) \mathbf{a} \\ &= \left(\mathbf{Q}_s - \mathbf{R}_s \left((\mathbf{R}_s^T \mathbf{K}_s \mathbf{R}_s)^{-1} \mathbf{R}_s^T \mathbf{K}_s \mathbf{Q}_s \right) \right) \mathbf{a} \\ &= \mathbf{F} \mathbf{a} \end{aligned} \quad (9.18)$$

The Craig-Bampton transformation \mathbf{F} with dimensions $(n \times n_b)$, can be expressed as

$$\mathbf{F} = \begin{bmatrix} \mathbf{I}_{bb} \\ \mathbf{\Phi}^c \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{bb} \\ -\mathbf{K}_{ii}^{-1} \mathbf{K}_{ib} \end{bmatrix} \quad (9.19)$$

9.2.3 Finding the matrix \mathbf{E} for undamped substructure systems

The second part of Craig-Bampton transformations \mathbf{E} is defined as substructure normal modes $\mathbf{\Phi}^n$. These modes are defined as the normal modes of the substructure for interior degrees of freedom relative to restrained boundaries.

$$\mathbf{E} = \begin{bmatrix} \mathbf{0} \\ \Phi^n \end{bmatrix} \quad (9.20)$$

Φ^n represents the normal modes of the constrained substructure. The \mathbf{E} part does not have its dimensions fixed. If the system has n degrees of freedom and n_b inputs (and outputs), then the maximum possible dimensions of \mathbf{E} are $(n \times (n - n_b))$. We choose how many columns to keep in \mathbf{E} . In a serious implementation of Craig-Bampton, one would make an initial choice arbitrarily (using a conservative over-estimate) and then potentially reduce the number of columns retained. The \mathbf{E} matrix is formed as

$$\mathbf{E} = \mathbf{R}_s \Phi^n \quad (9.21)$$

The generalised eigenvalue problem for the interior coordinates is defined by the matrices $\mathbf{M}_{ii} = \mathbf{R}^T \mathbf{M}_s \mathbf{R}$ and $\mathbf{K}_{ii} = \mathbf{R}^T \mathbf{K}_s \mathbf{R}$. Then, matrix Φ^n comprises sets of eigenvectors corresponding to the p lowest eigenvalues where p is some number not exceeding $(n - n_b)$.

$$(\mathbf{K}_{ii} - \omega_p^2 \mathbf{M}_{ii}) \phi_p = 0 \quad p = 1, 2, \dots, n - n_b \quad (9.22)$$

The eigenvectors of equation (9.22) are the normal modes of the constrained substructure and satisfy the following equations for the interior degrees of freedom.

$$\Phi^{nT} \mathbf{K}_{ii} \Phi^n = \mathbf{K}_{Dii} \quad , \quad \Phi^{nT} \mathbf{M}_{ii} \Phi^n = \mathbf{M}_{Dii} \quad (9.23)$$

The substructure normal modes or eigenvectors diagonalise the interior degrees of freedom matrices. The matrices $\{\mathbf{M}_{Dii}, \mathbf{K}_{Dii}\}$ in equation (9.23) are diagonal. If the mode shapes are mass normalized, then $\mathbf{M}_{Dii} = \mathbf{I}_{ii}$ and $\mathbf{K}_{Dii} = \Lambda_{ii}$ are diagonal. The Craig-Bampton transformations \mathbf{E} with dimensions $(n \times (n - n_b))$, can be expressed as

$$\mathbf{E} = \begin{bmatrix} \mathbf{0} \\ \Phi^n \end{bmatrix} \quad (9.24)$$

9.2.4 Substructure model matrices for undamped systems

The Craig-Bampton transformation can be written as

$$\mathbf{T}_{cb} = \begin{bmatrix} \mathbf{I}_{bb} & \mathbf{0} \\ -\mathbf{K}_{ii}^{-1}\mathbf{K}_{ib} & \Phi^n \end{bmatrix} \quad (9.25)$$

The Craig-Bampton transformation \mathbf{T}_{cb} transforms the substructure matrices into new system matrices. The substructure mass and stiffness matrices $\mathbf{M}_{cb}, \mathbf{K}_{cb}$ are defined as

$$\mathbf{M}_{cb} = \mathbf{T}_{cb}^T \mathbf{M}_s \mathbf{T}_{cb} = \begin{bmatrix} \mathbf{M}_{nbb} & \mathbf{M}_{nbi} \\ \mathbf{M}_{nib} & \mathbf{M}_{nii} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{nbb} & \mathbf{M}_{nbi} \\ \mathbf{M}_{nib} & \mathbf{I}_{ii} \end{bmatrix} \quad (9.26)$$

$$\mathbf{K}_{cb} = \mathbf{T}_{cb}^T \mathbf{K}_s \mathbf{T}_{cb} = \begin{bmatrix} \mathbf{K}_{nbb} & \mathbf{K}_{nbi} \\ \mathbf{K}_{nib} & \mathbf{K}_{nii} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{nbb} & \mathbf{0} \\ \mathbf{0} & \Lambda_{ii} \end{bmatrix} \quad (9.27)$$

where

$$\begin{aligned} \mathbf{M}_{nbb} &= \mathbf{M}_{bb} - \mathbf{M}_{bi} \mathbf{K}_{ii}^{-1} \mathbf{K}_{ib} - \mathbf{K}_{bi} \mathbf{K}_{ii}^{-1} (\mathbf{M}_{ib} - \mathbf{M}_{ii} \mathbf{K}_{ii}^{-1} \mathbf{K}_{ib}) \\ \mathbf{M}_{nbi} &= \mathbf{M}_{bi} \Phi^n - \mathbf{K}_{bi} \mathbf{K}_{ii}^{-1} \mathbf{M}_{ii} \Phi^n \\ \mathbf{M}_{nib} &= \Phi^{nT} \mathbf{M}_{ib} - \Phi^{nT} \mathbf{M}_{ii} \mathbf{K}_{ii}^{-1} \mathbf{K}_{ib} \\ \mathbf{M}_{nii} &= \mathbf{M}_{Dii} = \Phi^{nT} \mathbf{M}_{ii} \Phi^n = \mathbf{I}_{ii} \end{aligned} \quad (9.28)$$

$$\begin{aligned} \mathbf{K}_{nbb} &= \mathbf{K}_{bb} - \mathbf{K}_{bi} \mathbf{K}_{ii}^{-1} \mathbf{K}_{ib} \\ \mathbf{K}_{nbi} &= \mathbf{0} \\ \mathbf{K}_{nib} &= \mathbf{0} \\ \mathbf{K}_{nii} &= \mathbf{K}_{Dii} = \Phi_L^{nT} \mathbf{K}_{ii} \Phi_R^n = \Lambda_{ii} \end{aligned} \quad (9.29)$$

The second part of the Craig-Bampton transformation \mathbf{E} diagonalises the substructure interior coordinate matrices $\{\mathbf{M}_{ii}, \mathbf{K}_{ii}\}$ which represent interior coordinates (or eliminated degrees of freedom) such that

$$\begin{aligned} \mathbf{E}^T \mathbf{M}_s \mathbf{E} &= \Phi^{nT} \mathbf{M}_{ii} \Phi^n = \mathbf{M}_{Dii} \\ \mathbf{E}^T \mathbf{K}_s \mathbf{E} &= \Phi^{nT} \mathbf{K}_{ii} \Phi^n = \mathbf{K}_{Dii} \end{aligned} \quad (9.30)$$

The first part of the Craig Bampton transformation \mathbf{F} is used also to identify boundary coordinates (or reduced degrees of freedom) such that

$$\mathbf{F}^T \mathbf{K}_s \mathbf{F} = \mathbf{K}_{nbb} \quad (9.31)$$

Equation (9.19) shows that the constrain modes have stiffness orthogonal to all of the substructure normal modes, that is

$$\mathbf{E}^T \mathbf{K}_s \mathbf{F} = 0 \quad (9.32)$$

9.3 Damped substructure

The equation of motion for an n DoFs damped symmetric substructure can be expressed as

$$\begin{aligned} \mathbf{f}_s &= \mathbf{S}_s \mathbf{u}_s \\ \mathbf{M}_s \ddot{\mathbf{q}}_s(t) + \mathbf{D}_s \dot{\mathbf{q}}_s(t) + \mathbf{K}_s \mathbf{q}_s(t) &= \mathbf{f}_s(t) \\ \mathbf{y}_s &= \mathbf{S}_s^T \mathbf{q}_s \end{aligned} \quad (9.33)$$

where \mathbf{S}_s represents selection matrix at the terminals output and input n_b DoFs. The matrix \mathbf{S}_s is defined as $\begin{bmatrix} \mathbf{I}_{(n_b \times n_b)} & \mathbf{0}_{(n-n_b)} \end{bmatrix}$. Based on the division of the boundary coordinates and interior coordinates, equation (9.33) can be expressed as

$$\begin{bmatrix} \mathbf{M}_{bb} & \mathbf{M}_{bi} \\ \mathbf{M}_{ib} & \mathbf{M}_{ii} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}}_b \\ \ddot{\mathbf{q}}_i \end{Bmatrix} + \begin{bmatrix} \mathbf{D}_{bb} & \mathbf{D}_{bi} \\ \mathbf{D}_{ib} & \mathbf{D}_{ii} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{q}}_b \\ \dot{\mathbf{q}}_i \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{bb} & \mathbf{K}_{bi} \\ \mathbf{K}_{ib} & \mathbf{K}_{ii} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_b \\ \mathbf{q}_i \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_b \\ \mathbf{f}_i \end{Bmatrix} \quad (9.34)$$

It is possible in this case to write the dynamic system equation (9.33) in state space as

$$\begin{bmatrix} \mathbf{K}_s & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_s \end{bmatrix} \begin{Bmatrix} \mathbf{q}_s \\ \mathbf{r}_s \end{Bmatrix} + \begin{bmatrix} \mathbf{D}_s & \mathbf{M}_s \\ \mathbf{M}_s & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{q}}_s \\ \dot{\mathbf{r}}_s \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_s \\ \mathbf{0} \end{Bmatrix} \quad (9.35)$$

$$\begin{bmatrix} \mathbf{0} & \mathbf{K}_s \\ \mathbf{K}_s & \mathbf{D}_s \end{bmatrix} \begin{Bmatrix} \mathbf{q}_s \\ \mathbf{r}_s \end{Bmatrix} - \begin{bmatrix} \mathbf{K}_s & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_s \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{q}}_s \\ \dot{\mathbf{r}}_s \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{f}_s \end{Bmatrix} \quad (9.36)$$

Based on the division of the boundary coordinates and interior coordinates, equations (9.35) and (9.36) can be expressed in state space as

$$\begin{bmatrix} \begin{bmatrix} \mathbf{K}_{bb} & \mathbf{K}_{bi} \\ \mathbf{K}_{ib} & \mathbf{K}_{ii} \end{bmatrix} & \mathbf{0} \\ \mathbf{0} & -\begin{bmatrix} \mathbf{M}_{bb} & \mathbf{M}_{bi} \\ \mathbf{M}_{ib} & \mathbf{M}_{ii} \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_b \\ \mathbf{q}_i \\ \mathbf{r}_b \\ \mathbf{r}_i \end{Bmatrix} + \begin{bmatrix} \begin{bmatrix} \mathbf{D}_{bb} & \mathbf{D}_{bi} \\ \mathbf{D}_{ib} & \mathbf{D}_{ii} \end{bmatrix} & \begin{bmatrix} \mathbf{M}_{bb} & \mathbf{M}_{bi} \\ \mathbf{M}_{ib} & \mathbf{M}_{ii} \end{bmatrix} \\ \begin{bmatrix} \mathbf{M}_{bb} & \mathbf{M}_{bi} \\ \mathbf{M}_{ib} & \mathbf{M}_{ii} \end{bmatrix} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{q}}_b \\ \dot{\mathbf{q}}_i \\ \dot{\mathbf{r}}_b \\ \dot{\mathbf{r}}_i \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_b \\ \mathbf{f}_i \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \quad (9.37)$$

where $\mathbf{r}_s = \dot{\mathbf{q}}_s$, $\dot{\mathbf{r}}_s = \ddot{\mathbf{q}}_s$, $\mathbf{r}_s = \begin{Bmatrix} \mathbf{r}_b \\ \mathbf{r}_i \end{Bmatrix}$ and $\mathbf{q}_s = \begin{Bmatrix} \mathbf{q}_b \\ \mathbf{q}_i \end{Bmatrix}$

$$\underline{\mathbf{M}}_s = \begin{bmatrix} \mathbf{0} & \mathbf{K}_s \\ \mathbf{K}_s & \mathbf{D}_s \end{bmatrix} =: \begin{bmatrix} \mathbf{0} & \begin{bmatrix} \mathbf{K}_{bb} & \mathbf{K}_{bi} \\ \mathbf{K}_{ib} & \mathbf{K}_{ii} \end{bmatrix} \\ \begin{bmatrix} \mathbf{K}_{bb} & \mathbf{K}_{bi} \\ \mathbf{K}_{ib} & \mathbf{K}_{ii} \end{bmatrix} & \begin{bmatrix} \mathbf{D}_{bb} & \mathbf{D}_{bi} \\ \mathbf{D}_{ib} & \mathbf{D}_{ii} \end{bmatrix} \end{bmatrix} \quad (9.38)$$

$$\underline{\mathbf{D}}_s = \begin{bmatrix} \mathbf{K}_s & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_s \end{bmatrix} =: \begin{bmatrix} \begin{bmatrix} \mathbf{K}_{bb} & \mathbf{K}_{bi} \\ \mathbf{K}_{ib} & \mathbf{K}_{ii} \end{bmatrix} & \mathbf{0} \\ \mathbf{0} & -\begin{bmatrix} \mathbf{M}_{bb} & \mathbf{M}_{bi} \\ \mathbf{M}_{ib} & \mathbf{M}_{ii} \end{bmatrix} \end{bmatrix} \quad (9.39)$$

$$\underline{\mathbf{K}}_s = \begin{bmatrix} -\mathbf{D}_s & -\mathbf{M}_s \\ -\mathbf{M}_s & \mathbf{0} \end{bmatrix} =: \begin{bmatrix} -\begin{bmatrix} \mathbf{D}_{bb} & \mathbf{D}_{bi} \\ \mathbf{D}_{ib} & \mathbf{D}_{ii} \end{bmatrix} & -\begin{bmatrix} \mathbf{M}_{bb} & \mathbf{M}_{bi} \\ \mathbf{M}_{ib} & \mathbf{M}_{ii} \end{bmatrix} \\ -\begin{bmatrix} \mathbf{M}_{bb} & \mathbf{M}_{bi} \\ \mathbf{M}_{ib} & \mathbf{M}_{ii} \end{bmatrix} & \mathbf{0} \end{bmatrix} \quad (9.40)$$

The matrices $\{\underline{\mathbf{M}}_s, \underline{\mathbf{D}}_s, \underline{\mathbf{K}}_s\}$ define Lancaster augmented matrices *LAMs* as introduced earlier in Chapter 2. Rearranging *LAMs* in equations (9.38)-(9.40) for the boundary coordinates and interior coordinates, equation (9.37) can be written in the form

$$\begin{bmatrix} \begin{bmatrix} \mathbf{K}_{bb} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{bb} \end{bmatrix} & \begin{bmatrix} \mathbf{K}_{bi} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{bi} \end{bmatrix} \\ \begin{bmatrix} \mathbf{K}_{ib} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{ib} \end{bmatrix} & \begin{bmatrix} \mathbf{K}_{ii} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{ii} \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_b \\ \mathbf{r}_b \\ \mathbf{q}_i \\ \mathbf{r}_i \end{Bmatrix} + \begin{bmatrix} \begin{bmatrix} \mathbf{D}_{bb} & \mathbf{M}_{bb} \\ \mathbf{M}_{bb} & \mathbf{0} \end{bmatrix} & \begin{bmatrix} \mathbf{D}_{bi} & \mathbf{M}_{bi} \\ \mathbf{M}_{bi} & \mathbf{0} \end{bmatrix} \\ \begin{bmatrix} \mathbf{D}_{ib} & \mathbf{M}_{ib} \\ \mathbf{M}_{ib} & \mathbf{0} \end{bmatrix} & \begin{bmatrix} \mathbf{D}_{ii} & \mathbf{M}_{ii} \\ \mathbf{M}_{ii} & \mathbf{0} \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{q}}_b \\ \dot{\mathbf{r}}_b \\ \dot{\mathbf{q}}_i \\ \dot{\mathbf{r}}_i \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_b \\ \mathbf{0} \\ \mathbf{f}_i \\ \mathbf{0} \end{Bmatrix} \quad (9.41)$$

where

$$\underline{\mathbf{M}}_s = \begin{bmatrix} \mathbf{0} & \mathbf{K}_{bb} & \mathbf{0} & \mathbf{K}_{bi} \\ \mathbf{K}_{bb} & \mathbf{D}_{bb} & \mathbf{K}_{bi} & \mathbf{D}_{bi} \\ \mathbf{0} & \mathbf{K}_{ib} & \mathbf{0} & \mathbf{K}_{ii} \\ \mathbf{K}_{ib} & \mathbf{D}_{ib} & \mathbf{K}_{ii} & \mathbf{D}_{ii} \end{bmatrix} \quad (9.42)$$

$$\underline{\underline{\mathbf{D}}}_s = \begin{bmatrix} \mathbf{K}_{bb} & \mathbf{0} & \mathbf{K}_{bi} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{bb} & \mathbf{0} & -\mathbf{M}_{bi} \\ \mathbf{K}_{ib} & \mathbf{0} & \mathbf{K}_{ii} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{ib} & \mathbf{0} & -\mathbf{M}_{ii} \end{bmatrix} \quad (9.43)$$

$$\underline{\underline{\mathbf{K}}}_s = \begin{bmatrix} \mathbf{D}_{bb} & -\mathbf{M}_{bb} & -\mathbf{D}_{bi} & -\mathbf{M}_{bi} \\ -\mathbf{M}_{bb} & \mathbf{0} & -\mathbf{M}_{bi} & \mathbf{0} \\ -\mathbf{D}_{ib} & -\mathbf{M}_{ib} & -\mathbf{D}_{ii} & -\mathbf{M}_{ii} \\ -\mathbf{M}_{ib} & \mathbf{0} & -\mathbf{M}_{ii} & \mathbf{0} \end{bmatrix} \quad (9.44)$$

Simplifying equation (9.41) as

$$\begin{bmatrix} \underline{\underline{\mathbf{D}}}_{bb} & \underline{\underline{\mathbf{D}}}_{bi} \\ \underline{\underline{\mathbf{D}}}_{ib} & \underline{\underline{\mathbf{D}}}_{ii} \end{bmatrix} \begin{Bmatrix} \underline{\underline{\mathbf{q}}}_b \\ \underline{\underline{\mathbf{q}}}_i \end{Bmatrix} + \begin{bmatrix} \underline{\underline{\mathbf{K}}}_{bb} & \underline{\underline{\mathbf{K}}}_{bi} \\ \underline{\underline{\mathbf{K}}}_{ib} & \underline{\underline{\mathbf{K}}}_{ii} \end{bmatrix} \begin{Bmatrix} \underline{\underline{\dot{\mathbf{q}}}}_b \\ \underline{\underline{\dot{\mathbf{q}}}}_i \end{Bmatrix} = \begin{Bmatrix} \underline{\underline{\mathbf{f}}}_b \\ \underline{\underline{\mathbf{f}}}_i \end{Bmatrix} \quad (9.45)$$

Equation (9.45) can be written in the form

$$\begin{aligned} \underline{\underline{\mathbf{f}}}_s &= \underline{\underline{\mathbf{S}}}_s \underline{\underline{\mathbf{u}}} \\ \underline{\underline{\mathbf{D}}}_s \underline{\underline{\mathbf{q}}}_s + \underline{\underline{\mathbf{K}}}_s \underline{\underline{\dot{\mathbf{q}}}}_s &= \underline{\underline{\mathbf{f}}}_s \\ \underline{\underline{\mathbf{y}}}_s &= \underline{\underline{\mathbf{S}}}_s^T \underline{\underline{\mathbf{q}}}_s \end{aligned} \quad (9.46)$$

Underline ($\underline{\quad}$) represents double dimensions ($2n$). $\underline{\underline{\mathbf{q}}}_s = \begin{Bmatrix} \underline{\underline{\mathbf{q}}}_b \\ \underline{\underline{\mathbf{q}}}_i \end{Bmatrix}$, $\underline{\underline{\mathbf{q}}}_b = \begin{Bmatrix} \underline{\underline{\mathbf{q}}}_b \\ \underline{\underline{\mathbf{r}}}_b \end{Bmatrix}$ and $\underline{\underline{\mathbf{q}}}_i = \begin{Bmatrix} \underline{\underline{\mathbf{q}}}_i \\ \underline{\underline{\mathbf{r}}}_i \end{Bmatrix}$.

9.4 Craig-Bampton transformations for damped second order systems

This section extends the developed Craig-Bampton transformations (CBTs) for undamped systems in section 9.2 to damped systems. The Craig-Bampton transformation in state space is defined as

$$\underline{\underline{\mathbf{q}}}_s = \underline{\underline{\mathbf{T}}}_{Rcb} \underline{\underline{\mathbf{q}}}_{sn} \quad (9.47)$$

$$\underline{\underline{\mathbf{f}}}_{sn} = \underline{\underline{\mathbf{T}}}_{Lcb}^T \underline{\underline{\mathbf{f}}}_s \quad (9.48)$$

Craig-Bampton transformation matrix, $\underline{\underline{\mathbf{T}}}_{cb}$ is divided into two separate components

$$\underline{\underline{\mathbf{T}}}_{cb} = [\underline{\underline{\mathbf{F}}} \quad \underline{\underline{\mathbf{E}}}] \quad (9.49)$$

9.4.1 Finding the matrix $\underline{\mathbf{F}}$ for damped substructure systems in state space

By comparing equation (9.46) with equation (9.1) for undamped systems then,

$$\underline{\mathbf{a}} = \underline{\mathbf{S}}_s^T \underline{\mathbf{q}}_s \quad (9.50)$$

where $\underline{\mathbf{a}}$ represents a double dimension $(2n_b \times 1)$, arbitrary vector describing the prescribed deflections and velocities at the terminals. The space of all vectors, $\underline{\mathbf{q}}_s$ (double dimension), which can satisfy equation (9.50) is given by

$$\underline{\mathbf{q}}_s = \underline{\mathbf{S}}_s \left(\underline{\mathbf{S}}_s^T \underline{\mathbf{S}}_s \right)^{-1} \underline{\mathbf{a}} + \underline{\mathbf{R}}_s \underline{\mathbf{b}} =: \underline{\mathbf{Q}}_s \underline{\mathbf{a}} + \underline{\mathbf{R}}_s \underline{\mathbf{b}} \quad (9.51)$$

where $\underline{\mathbf{b}}$ represents some arbitrary vector, has double dimensions $(2(n-n_b) \times 1)$.

Then, the matrix $\underline{\mathbf{S}}_s$ is defined as $\left[\underline{\mathbf{I}}_{(2n_b \times 2n_b)} ; \underline{\mathbf{0}}_{(2n-2n_b)} \right]$ and the matrix, $\underline{\mathbf{R}}_s$ $(2n \times (2n-2n_b))$, can be found such that : $\left[\underline{\mathbf{S}}_s \quad \underline{\mathbf{R}}_s \right]$ is a square invertible matrix.

Following the same procedure described in section 9.2.2, the potential energy and kinetic energy $\mathfrak{P}\mathfrak{K}_E$ are expressed as

$$\mathfrak{P}\mathfrak{K}_E = \underline{\mathbf{q}}_s^T \underline{\mathbf{D}}_s \underline{\mathbf{q}}_s = \begin{bmatrix} \underline{\mathbf{a}}^T & \underline{\mathbf{b}}^T \end{bmatrix} \begin{bmatrix} \underline{\mathbf{D}}_{bb} & \underline{\mathbf{D}}_{bi} \\ \underline{\mathbf{D}}_{ib} & \underline{\mathbf{D}}_{ii} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{a}} \\ \underline{\mathbf{b}} \end{bmatrix} \quad (9.52)$$

where

$$\begin{aligned} \underline{\mathbf{D}}_{bb} &= \underline{\mathbf{Q}}_s^T \underline{\mathbf{D}}_s \underline{\mathbf{Q}}_s, & \underline{\mathbf{D}}_{bi} &= \underline{\mathbf{Q}}_s^T \underline{\mathbf{D}}_s \underline{\mathbf{R}}_s \\ \underline{\mathbf{D}}_{ib} &= \underline{\mathbf{R}}_s^T \underline{\mathbf{D}}_s \underline{\mathbf{Q}}_s, & \underline{\mathbf{D}}_{ii} &= \underline{\mathbf{R}}_s^T \underline{\mathbf{D}}_s \underline{\mathbf{R}}_s \end{aligned} \quad (9.53)$$

We note that the matrix $\underline{\mathbf{D}}_s$ is not generally positive definite. Following Guyan's logic [52], the energy $\mathfrak{P}\mathfrak{K}_E$ in equation (9.52) are minimised by choosing:

$$\underline{\mathbf{b}} = - \left(\underline{\mathbf{D}}_{ii}^{-1} \underline{\mathbf{D}}_{ib} \right) \underline{\mathbf{a}} \quad (9.54)$$

Substitute all of this back into equation (9.51) to find

$$\begin{aligned}
\underline{\mathbf{q}}_s &= \left(\underline{\mathbf{Q}}_s - \underline{\mathbf{R}}_s \left(\underline{\mathbf{D}}_{ii}^{-1} \underline{\mathbf{D}}_{ib} \right) \right) \underline{\mathbf{a}} \\
&= \left(\underline{\mathbf{Q}}_s - \underline{\mathbf{R}}_s \left(\left(\underline{\mathbf{R}}_s^T \underline{\mathbf{D}}_s \underline{\mathbf{R}}_s \right)^{-1} \left(\underline{\mathbf{R}}_s^T \underline{\mathbf{D}}_s \underline{\mathbf{Q}}_s \right) \right) \right) \underline{\mathbf{a}} \\
&= \underline{\mathbf{F}} \underline{\mathbf{a}}
\end{aligned} \tag{9.55}$$

The Craig-Bampton transformation $\underline{\mathbf{F}}$ in state space has dimensions $(2n \times 2n_b)$ and can be expressed as

$$\underline{\mathbf{F}} = \begin{bmatrix} \mathbf{I}_{(nb \times nb)} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{(nb \times nb)} \\ -\mathbf{K}_{ii}^{-1} \mathbf{K}_{ib} & \mathbf{0}_{((n-nb) \times nb)} \\ \mathbf{0}_{((n-nb) \times nb)} & -\mathbf{M}_{ii}^{-1} \mathbf{M}_{ib} \end{bmatrix} \tag{9.56}$$

The matrices $(\mathbf{K}_{ii}^{-1} \mathbf{K}_{ib})$ and $(\mathbf{M}_{ii}^{-1} \mathbf{M}_{ib})$ have dimensions $((n - n_b) \times n_b)$.

9.4.2 Finding the matrix $\underline{\mathbf{E}}$ for damped substructure systems in state space

The eigenvalue problem for the interior coordinates is defined by the matrices

$$\underline{\mathbf{D}}_{ii} = \underline{\mathbf{R}}^T \underline{\mathbf{D}}_s \underline{\mathbf{R}} \text{ and } \underline{\mathbf{K}}_{ii} = \underline{\mathbf{R}}^T \underline{\mathbf{K}}_s \underline{\mathbf{R}}.$$

$$(\underline{\mathbf{D}}_{ii} - \lambda_p \underline{\mathbf{K}}_{ii}) \underline{\phi}_p = 0 \quad p = 1, 2, \dots, n - n_b \tag{9.57}$$

The $\underline{\phi}_p$ comprises a pair of eigenvectors in the modal matrix $\underline{\Phi}^n$. The maximum number of eigenvalues are not exceeding $(2n - 2n_b)$. The modal matrix $\underline{\Phi}^n$ diagonalises the interior matrices $\{\underline{\mathbf{K}}_{ii}, \underline{\mathbf{D}}_{ii}, \underline{\mathbf{M}}_{ii}\}$ and satisfies the following equations

$$\begin{aligned}
\underline{\mathbf{M}}_{Dii} &= \underline{\Phi}^{nT} \underline{\mathbf{M}}_{ii} \underline{\Phi}^n \\
\underline{\mathbf{D}}_{Dii} &= \underline{\Phi}^{nT} \underline{\mathbf{D}}_{ii} \underline{\Phi}^n \\
\underline{\mathbf{K}}_{Dii} &= \underline{\Phi}^{nT} \underline{\mathbf{K}}_{ii} \underline{\Phi}^n
\end{aligned} \tag{9.58}$$

In equation (9.58), the matrices $\{\underline{\mathbf{K}}_{Dii}, \underline{\mathbf{D}}_{Dii}, \underline{\mathbf{M}}_{Dii}\}$ are diagonal. The Craig-Bampton transformation $\underline{\mathbf{E}}$ in state space with dimensions $(2n \times (2n - 2n_b))$ can be expressed as

$$\underline{\mathbf{E}} = \underline{\mathbf{R}}_s \underline{\Phi}^n = \begin{bmatrix} \mathbf{0}_{(2nb \times (2n - n_b))} \\ \underline{\Phi}^n_{(2(n - nb) \times 2(n - n_b))} \end{bmatrix} \quad (9.59)$$

9.4.3 Substructure model matrices for damped systems

The Craig-Bampton transformations $\{\underline{\mathbf{T}}_{cb}\}$ for damped symmetric substructure systems in state space can be written as

$$\underline{\mathbf{T}}_{cb} = \begin{bmatrix} \mathbf{I}_{(nb \times nb)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{(nb \times nb)} & \mathbf{0} \\ \hline -\mathbf{K}_{ii}^{-1} \mathbf{K}_{ib} & \mathbf{0}_{((n - nb) \times nb)} & \mathbf{0} \\ \mathbf{0}_{((n - nb) \times nb)} & -\mathbf{M}_{ii}^{-1} \mathbf{M}_{ib} & \underline{\Phi}^n_{(2(n - nb) \times 2(n - n_b))} \end{bmatrix} \quad (9.60)$$

The substructure model matrices $(\underline{\mathbf{M}}_{cb}, \underline{\mathbf{D}}_{cb}, \underline{\mathbf{K}}_{cb})$ using equations are defined as

$$\underline{\mathbf{M}}_{cb} = \underline{\mathbf{T}}_{cb}^T \underline{\mathbf{M}}_s \underline{\mathbf{T}}_{cb} = \begin{bmatrix} \underline{\mathbf{M}}_{nbb} & \underline{\mathbf{M}}_{nbi} \\ \underline{\mathbf{M}}_{nib} & \underline{\mathbf{M}}_{nii} \end{bmatrix} \quad (9.61)$$

$$\underline{\mathbf{D}}_{cb} = \underline{\mathbf{T}}_{cb}^T \underline{\mathbf{D}}_s \underline{\mathbf{T}}_{cb} = \begin{bmatrix} \underline{\mathbf{D}}_{nbb} & \underline{\mathbf{D}}_{nbi} \\ \underline{\mathbf{D}}_{nib} & \underline{\mathbf{D}}_{nii} \end{bmatrix} \quad (9.62)$$

$$\underline{\mathbf{K}}_{cb} = \underline{\mathbf{T}}_{cb}^T \underline{\mathbf{K}}_s \underline{\mathbf{T}}_{cb} = \begin{bmatrix} \underline{\mathbf{K}}_{nbb} & \underline{\mathbf{K}}_{nbi} \\ \underline{\mathbf{K}}_{nib} & \underline{\mathbf{K}}_{nii} \end{bmatrix} \quad (9.63)$$

The Craig-Bampton transformation $\underline{\mathbf{E}}$ diagonalises the substructure interior coordinates matrices $\{\underline{\mathbf{M}}_{ii}, \underline{\mathbf{D}}_{ii}, \underline{\mathbf{K}}_{ii}\}$.

$$\begin{aligned} \underline{\mathbf{M}}_{Dii} &= \underline{\mathbf{E}}^T \underline{\mathbf{M}}_s \underline{\mathbf{E}} \\ \underline{\mathbf{D}}_{Dii} &= \underline{\mathbf{E}}^T \underline{\mathbf{D}}_s \underline{\mathbf{E}} \\ \underline{\mathbf{K}}_{Dii} &= \underline{\mathbf{E}}^T \underline{\mathbf{K}}_s \underline{\mathbf{E}} \end{aligned} \quad (9.64)$$

Equation (9.56) shows that the constrain modes have mass and stiffness orthogonal to all of the substructure normal modes, that are

$$\underline{\mathbf{E}}^T \underline{\mathbf{D}}_s \underline{\mathbf{F}} = 0 \quad , \quad \underline{\mathbf{F}}^T \underline{\mathbf{D}}_s \underline{\mathbf{E}} = 0 \quad (9.65)$$

Craig-Bampton transformation $\underline{\mathbf{F}}$ in state space gives

$$\underline{\mathbf{F}}^T \underline{\mathbf{D}}_s \underline{\mathbf{F}} = \underline{\mathbf{D}}_{nbb} = \begin{bmatrix} \mathbf{K}_{bb} - \mathbf{K}_{bi} \mathbf{K}_{ii}^{-1} \mathbf{K}_{ib} & 0 \\ 0 & -\mathbf{M}_{bb} + \mathbf{M}_{bi} \mathbf{M}_{ii}^{-1} \mathbf{M}_{ib} \end{bmatrix} \quad (9.66)$$

9.5 Generalised Craig-Bampton Transformations

In a general case for non-symmetric matrices, the right and left Craig-Bampton transformations $\{\underline{\mathbf{T}}_{Rcb}, \underline{\mathbf{T}}_{Lcb}\}$ are expressed each in two parts as

$$\underline{\mathbf{T}}_{Rcb} = \begin{bmatrix} \underline{\mathbf{F}}_R & \underline{\mathbf{E}}_R \end{bmatrix} \quad , \quad \underline{\mathbf{T}}_{Lcb} = \begin{bmatrix} \underline{\mathbf{F}}_L & \underline{\mathbf{E}}_L \end{bmatrix} \quad (9.67)$$

where

$$\underline{\mathbf{F}}_R = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \\ -(\mathbf{K}_{ii}^{-1} \mathbf{K}_{ib}) & 0 \\ 0 & -(\mathbf{M}_{ii}^{-1} \mathbf{M}_{ib}) \end{bmatrix} \quad , \quad \underline{\mathbf{F}}_L = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \\ -(\mathbf{K}_{bi} \mathbf{K}_{ii}^{-1})^T & 0 \\ 0 & -(\mathbf{M}_{bi} \mathbf{M}_{ii}^{-1})^T \end{bmatrix} \quad (9.68)$$

The matrices $\underline{\mathbf{E}}_R$ and $\underline{\mathbf{E}}_L$ can be solved from the eigenvalue problem of equation (9.57) and it's transpose defined by the matrices $\underline{\mathbf{D}}_{ii}^T$ and $\underline{\mathbf{K}}_{ii}^T$.

$$\underline{\mathbf{E}}_R = \underline{\mathbf{R}} \underline{\Phi}_R^n \quad , \quad \underline{\mathbf{E}}_L = \underline{\mathbf{R}} \underline{\Phi}_L^n \quad (9.69)$$

The matrices $\{\underline{\Phi}_R^n, \underline{\Phi}_L^n\}$ are chosen at the same eigenvalues of the interior coordinates. The eigenvalues and corresponding eigenvectors for the interior coordinates can be either real or complex conjugate or mix.

Real valued diagonalising SPEs $\{\underline{\mathbf{T}}_{Rii}, \underline{\mathbf{T}}_{Lii}\}$ for the interior coordinates can be calculated. The transformation $\{\underline{\mathbf{E}}_R, \underline{\mathbf{E}}_L\}$ for the interior coordinates in state space can be expressed in terms of diagonalising SPEs for the interior coordinates. The interested modes can be chosen by discarding most of high frequency interior modes.

$$\underline{\mathbf{E}}_R = \begin{bmatrix} \mathbf{0}_{(2nb(2n-n_b))} \\ \underline{\mathbf{T}}_{Rii}(2(n-nb) \times 2(n-n_b)) \end{bmatrix}, \quad \underline{\mathbf{E}}_L = \begin{bmatrix} \mathbf{0}_{(2nb(2n-n_b))} \\ \underline{\mathbf{T}}_{Lii}(2(n-nb) \times 2(n-n_b)) \end{bmatrix} \quad (9.70)$$

The right and left Craig-Bampton transformations $\{\underline{\mathbf{E}}_R, \underline{\mathbf{E}}_L\}$ satisfy equation (9.64).

The matrices $\{\underline{\mathbf{K}}_{Dii}, \underline{\mathbf{D}}_{Dii}, \underline{\mathbf{M}}_{Dii}\}$ have four block diagonal matrices. The following section presents Craig-Bampton transformations in SPEs form.

9.5.1 Craig Bampton transformations in SPEs form

Craig Bampton transformations for general damped substructure systems are extended to be as SPEs

$$\underline{\mathbf{T}}_{Rcb} = \begin{bmatrix} \underline{\mathbf{W}}_R & \underline{\mathbf{X}}_R \\ \underline{\mathbf{Y}}_R & \underline{\mathbf{Z}}_R \end{bmatrix}, \quad \underline{\mathbf{T}}_{Lcb} = \begin{bmatrix} \underline{\mathbf{W}}_L & \underline{\mathbf{X}}_L \\ \underline{\mathbf{Y}}_L & \underline{\mathbf{Z}}_L \end{bmatrix} \quad (9.71)$$

such that

$$\begin{bmatrix} \underline{\mathbf{W}}_L & \underline{\mathbf{X}}_L \\ \underline{\mathbf{Y}}_L & \underline{\mathbf{Z}}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \underline{\mathbf{K}}_s \\ \underline{\mathbf{K}}_s & \underline{\mathbf{D}}_s \end{bmatrix} \begin{bmatrix} \underline{\mathbf{W}}_R & \underline{\mathbf{X}}_R \\ \underline{\mathbf{Y}}_R & \underline{\mathbf{Z}}_R \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \underline{\mathbf{K}}_N \\ \underline{\mathbf{K}}_N & \underline{\mathbf{D}}_N \end{bmatrix} \quad (9.72)$$

$$\begin{bmatrix} \underline{\mathbf{W}}_L & \underline{\mathbf{X}}_L \\ \underline{\mathbf{Y}}_L & \underline{\mathbf{Z}}_L \end{bmatrix}^T \begin{bmatrix} \underline{\mathbf{K}}_s & \mathbf{0} \\ \mathbf{0} & -\underline{\mathbf{M}}_s \end{bmatrix} \begin{bmatrix} \underline{\mathbf{W}}_R & \underline{\mathbf{X}}_R \\ \underline{\mathbf{Y}}_R & \underline{\mathbf{Z}}_R \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{K}}_N & \mathbf{0} \\ \mathbf{0} & -\underline{\mathbf{M}}_N \end{bmatrix} \quad (9.73)$$

$$\begin{bmatrix} \underline{\mathbf{W}}_L & \underline{\mathbf{X}}_L \\ \underline{\mathbf{Y}}_L & \underline{\mathbf{Z}}_L \end{bmatrix}^T \begin{bmatrix} -\underline{\mathbf{D}}_s & -\underline{\mathbf{M}}_s \\ -\underline{\mathbf{M}}_s & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{\mathbf{W}}_R & \underline{\mathbf{X}}_R \\ \underline{\mathbf{Y}}_R & \underline{\mathbf{Z}}_R \end{bmatrix} = \begin{bmatrix} -\underline{\mathbf{D}}_N & -\underline{\mathbf{M}}_N \\ -\underline{\mathbf{M}}_N & \mathbf{0} \end{bmatrix} \quad (9.74)$$

more concisely

$$\underline{\mathbf{T}}_{Lcb}^T \underline{\mathbf{M}}_s \underline{\mathbf{T}}_{Rcb} = \underline{\mathbf{M}}_N \quad (9.75)$$

$$\underline{\mathbf{T}}_{Lcb}^T \underline{\mathbf{D}}_s \underline{\mathbf{T}}_{Rcb} = \underline{\mathbf{D}}_N \quad (9.76)$$

$$\underline{\mathbf{T}}_{Lcb}^T \underline{\mathbf{K}}_s \underline{\mathbf{T}}_{Rcb} = \underline{\mathbf{K}}_N \quad (9.77)$$

where

$$\underline{\mathbf{M}}_N = \begin{bmatrix} \mathbf{0} & \mathbf{K}_{Nbb} & \mathbf{0} & \mathbf{K}_{Nbi} \\ \mathbf{K}_{Nbb} & \mathbf{D}_{Nbb} & \mathbf{K}_{Nbi} & \mathbf{D}_{Nbi} \\ \mathbf{0} & \mathbf{K}_{Nib} & \mathbf{0} & \mathbf{K}_{Nii} \\ \mathbf{K}_{Nib} & \mathbf{D}_{Nib} & \mathbf{K}_{Nii} & \mathbf{D}_{Nii} \end{bmatrix} \quad (9.78)$$

$$\underline{\mathbf{D}}_N = \begin{bmatrix} \mathbf{K}_{Nbb} & \mathbf{0} & \mathbf{K}_{Nbi} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{Nbb} & \mathbf{0} & -\mathbf{M}_{Nbi} \\ \mathbf{K}_{Nib} & \mathbf{0} & \mathbf{K}_{Nii} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{Nib} & \mathbf{0} & -\mathbf{M}_{Nii} \end{bmatrix} \quad (9.79)$$

$$\underline{\mathbf{K}}_N = \begin{bmatrix} -\mathbf{D}_{Nbb} & -\mathbf{M}_{Nbb} & -\mathbf{D}_{Nbi} & -\mathbf{M}_{Nbi} \\ -\mathbf{M}_{Nbb} & \mathbf{0} & -\mathbf{M}_{Nbi} & \mathbf{0} \\ -\mathbf{D}_{Nib} & -\mathbf{M}_{Nib} & -\mathbf{D}_{Nii} & -\mathbf{M}_{Nii} \\ -\mathbf{M}_{Nib} & \mathbf{0} & -\mathbf{M}_{Nii} & \mathbf{0} \end{bmatrix} \quad (9.80)$$

The matrices $\{\underline{\mathbf{M}}_N, \underline{\mathbf{D}}_N, \underline{\mathbf{K}}_N\}$ have four blocks and each block has the same *LAMs* format. The matrices $\{\underline{\mathbf{M}}_{Nii}, \underline{\mathbf{D}}_{Nii}, \underline{\mathbf{K}}_{Nii}\}$ are all diagonal. The two parts of the Craig Bampton transformations $\{\underline{\mathbf{E}}_R, \underline{\mathbf{E}}_L\}$ and $\{\underline{\mathbf{F}}_R, \underline{\mathbf{F}}_L\}$ are calculated separately in the full size of the substructure. The reduced system matrices $\{\underline{\mathbf{M}}_{cb}, \underline{\mathbf{D}}_{cb}, \underline{\mathbf{K}}_{cb}\}$ are extracted from equations (9.78), (9.79) and (9.80)

$$\underline{\mathbf{M}}_{cb} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K}_{Nbb} & \mathbf{0} & \mathbf{K}_{Nbi} \\ \mathbf{K}_{Nbb} & \mathbf{D}_{Nbb} & \mathbf{K}_{Nbi} & \mathbf{D}_{Nbi} \\ \mathbf{0} & \mathbf{K}_{Nib} & \mathbf{0} & \mathbf{K}_{Nii} \\ \mathbf{K}_{Nib} & \mathbf{D}_{Nib} & \mathbf{K}_{Nii} & \mathbf{D}_{Nii} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_{cb} \\ \mathbf{K}_{cb} & \mathbf{D}_{cb} \end{bmatrix} \quad (9.81)$$

$$\underline{\mathbf{D}}_{cb} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_{Nbb} & \mathbf{0} & \mathbf{K}_{Nbi} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{Nbb} & \mathbf{0} & -\mathbf{M}_{Nbi} \\ \mathbf{K}_{Nib} & \mathbf{0} & \mathbf{K}_{Nii} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{Nib} & \mathbf{0} & -\mathbf{M}_{Nii} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{cb} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{cb} \end{bmatrix} \quad (9.82)$$

$$\underline{\mathbf{K}}_{cb} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}^T \begin{bmatrix} -\mathbf{D}_{Nbb} & -\mathbf{M}_{Nbb} & -\mathbf{D}_{Nbi} & -\mathbf{M}_{Nbi} \\ -\mathbf{M}_{Nbb} & \mathbf{0} & -\mathbf{M}_{Nbi} & \mathbf{0} \\ -\mathbf{D}_{Nib} & -\mathbf{M}_{Nib} & -\mathbf{D}_{Nii} & -\mathbf{M}_{Nii} \\ -\mathbf{M}_{Nib} & \mathbf{0} & -\mathbf{M}_{Nii} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} -\mathbf{D}_{cb} & -\mathbf{M}_{cb} \\ -\mathbf{M}_{cb} & \mathbf{0} \end{bmatrix} \quad (9.83)$$

9.5.2 Craig Bampton transforms $\{\underline{\mathbf{E}}_R, \underline{\mathbf{E}}_L\}$ in the full size of the substructure

The second part of the Craig Bampton transformations $\{\underline{\mathbf{E}}_R, \underline{\mathbf{E}}_L\}$ is extended to the full size of the substructure. These methods are simplified in a few points.

9.5.2.1 Diagonalising SPEs for the interior coordinates

The right and left diagonalising SPEs $\{\underline{\mathbf{T}}_{Rii}, \underline{\mathbf{T}}_{Lii}\}$ for the interior coordinate are

$$\underline{\mathbf{T}}_{Rii} = \begin{bmatrix} \mathbf{W}_{Rii} & \mathbf{X}_{Rii} \\ \mathbf{Y}_{Rii} & \mathbf{Z}_{Rii} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{R1} - \frac{1}{2} \mathbf{G}_{R1} \mathbf{D}_{Dii} & -\mathbf{G}_{R1} \mathbf{M}_{Dii} \\ \mathbf{G}_{R1} \mathbf{K}_{Dii} & \mathbf{F}_{R1} + \frac{1}{2} \mathbf{G}_{R1} \mathbf{D}_{Dii} \end{bmatrix} \quad (9.84)$$

$$\underline{\mathbf{T}}_{Lii} = \begin{bmatrix} \mathbf{W}_{Lii} & \mathbf{X}_{Lii} \\ \mathbf{Y}_{Lii} & \mathbf{Z}_{Lii} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{L1} - \frac{1}{2} \mathbf{G}_{L1} \mathbf{D}_{Dii}^T & -\mathbf{G}_{L1} \mathbf{M}_{Dii}^T \\ \mathbf{G}_{L1} \mathbf{K}_{Dii}^T & \mathbf{F}_{L1} + \frac{1}{2} \mathbf{G}_{L1} \mathbf{D}_{Dii}^T \end{bmatrix} \quad (9.85)$$

where $\mathbf{F}_{R1} \mathbf{G}_{L1}^T + \mathbf{G}_{R1} \mathbf{F}_{L1}^T = 0$

9.5.2.2 Extract $\mathbf{F}_{R2}, \mathbf{F}_{L2}, \mathbf{G}_{R2}, \mathbf{G}_{L2}$ from $\{\underline{\mathbf{T}}_{Rii}, \underline{\mathbf{T}}_{Lii}\}$ SPEs

The matrices $\mathbf{F}_{R2}, \mathbf{F}_{L2}, \mathbf{G}_{R2}, \mathbf{G}_{L2}$ are extracted from the inverse of the diagonalising transformations $\{\underline{\mathbf{T}}_{Rii}, \underline{\mathbf{T}}_{Lii}\}$ such that

$$\underline{\mathbf{T}}_{Rii}^{-1} = \begin{bmatrix} \mathbf{W}_{R2ii} & \mathbf{X}_{R2ii} \\ \mathbf{Y}_{R2ii} & \mathbf{Z}_{R2ii} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{R2} - \frac{1}{2}\mathbf{G}_{R2}\mathbf{D}_{ii} & -\mathbf{G}_{R2}\mathbf{M}_{ii} \\ \mathbf{G}_{R2}\mathbf{K}_{ii} & \mathbf{F}_{R2} + \frac{1}{2}\mathbf{G}_{R2}\mathbf{D}_{ii} \end{bmatrix} \quad (9.86)$$

$$\underline{\mathbf{T}}_{Lii}^{-1} = \begin{bmatrix} \mathbf{W}_{L2ii} & \mathbf{X}_{L2ii} \\ \mathbf{Y}_{L2ii} & \mathbf{Z}_{L2ii} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{L2} - \frac{1}{2}\mathbf{G}_{L2}\mathbf{D}_{ii}^T & -\mathbf{G}_{L2}\mathbf{M}_{ii}^T \\ \mathbf{G}_{L2}\mathbf{K}_{ii}^T & \mathbf{F}_{L2} + \frac{1}{2}\mathbf{G}_{L2}\mathbf{D}_{ii}^T \end{bmatrix} \quad (9.87)$$

Using the pseudo inverse for determining $\mathbf{G}_{R2}, \mathbf{G}_{L2}$

$$\mathbf{G}_{R2} = [\mathbf{X}_{RM} \quad \mathbf{X}_{RD} \quad \mathbf{X}_{RK}] [\mathbf{M}_{ii} \quad \mathbf{D}_{ii} \quad \mathbf{K}_{ii}]^+ \quad (9.88)$$

$$\mathbf{G}_{L2} = [\mathbf{X}_{LM} \quad \mathbf{X}_{LD} \quad \mathbf{X}_{LK}] [\mathbf{M}_{ii}^T \quad \mathbf{D}_{ii}^T \quad \mathbf{K}_{ii}^T]^+ \quad (9.89)$$

where superscript “+” denotes pseudo inverse. The matrices $\mathbf{X}_{RM}, \mathbf{X}_{RD}, \mathbf{X}_{RK}$ and $\mathbf{X}_{LM}, \mathbf{X}_{LD}, \mathbf{X}_{LK}$ are defined as

$$\mathbf{X}_{RM} = \mathbf{X}_{R2ii} \quad , \quad \mathbf{X}_{RD} = (\mathbf{Z}_{R2ii} - \mathbf{W}_{R2ii}) \quad , \quad \mathbf{X}_{RK} = \mathbf{Y}_{R2ii} \quad (9.90)$$

$$\mathbf{X}_{LM} = \mathbf{X}_{L2ii} \quad , \quad \mathbf{X}_{LD} = (\mathbf{Z}_{L2ii} - \mathbf{W}_{L2ii}) \quad , \quad \mathbf{X}_{LK} = \mathbf{Y}_{L2ii} \quad (9.91)$$

We note that

$$\mathbf{G}_{R1} = \mathbf{G}_{L2}^T \quad , \quad \mathbf{G}_{L1} = \mathbf{G}_{R2}^T \quad (9.92)$$

The values of $\mathbf{F}_{R2}, \mathbf{F}_{L2}$ are calculated

$$\mathbf{F}_{R2} = (\mathbf{W}_{R2ii} - \mathbf{Z}_{R2ii})/2 \quad , \quad \mathbf{F}_{L2} = (\mathbf{W}_{L2ii} - \mathbf{Z}_{L2ii})/2 \quad (9.93)$$

where $\mathbf{F}_{R2}\mathbf{G}_{L2}^T + \mathbf{G}_{R2}\mathbf{F}_{L2}^T = 0$.

9.5.2.3 Extending $\mathbf{F}_{R2}, \mathbf{F}_{L2}, \mathbf{G}_{R2}, \mathbf{G}_{L2}$ to the full size of the substructure

The matrices $\mathbf{F}_{R2}, \mathbf{F}_{L2}, \mathbf{G}_{R2}, \mathbf{G}_{L2}$ are extended to the full size of the substructure such that

$$\underline{\mathbf{T}}_R^{-1} = \begin{bmatrix} \mathbf{F}_{R2-} - \frac{1}{2} \mathbf{G}_{R2-} \mathbf{D}_s & -\mathbf{G}_{R2-} \mathbf{M}_s \\ \mathbf{G}_{R2-} \mathbf{K}_s & \mathbf{F}_{R2-} + \frac{1}{2} \mathbf{G}_{R2-} \mathbf{D}_s \end{bmatrix} \quad (9.94)$$

$$\underline{\mathbf{T}}_L^{-1} = \begin{bmatrix} \mathbf{F}_{L2-} - \frac{1}{2} \mathbf{G}_{L2-} \mathbf{D}_s^T & -\mathbf{G}_{L2-} \mathbf{M}_s^T \\ \mathbf{G}_{L2-} \mathbf{K}_s^T & \mathbf{F}_{L2-} + \frac{1}{2} \mathbf{G}_{L2-} \mathbf{D}_s^T \end{bmatrix} \quad (9.95)$$

where the matrices $\{\mathbf{F}_{R2-}, \mathbf{F}_{L2-}, \mathbf{G}_{R2-}, \mathbf{G}_{L2-}\}$ have dimensions $(n \times n)$. The right and left modified Craig Bampton transformations $\{\underline{\mathbf{E}}_R, \underline{\mathbf{E}}_L\}$ can be expressed as

$$\underline{\mathbf{T}}_R^{-1} = \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{T}_{Rib} & \mathbf{T}_{Rii} \end{bmatrix} = \left[\begin{array}{cc|cc} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \hline -\frac{1}{2} \mathbf{G}_{R2} \mathbf{D}_{ib} & -\mathbf{G}_{R2} \mathbf{M}_{ib} & \mathbf{F}_{R2} - \frac{1}{2} \mathbf{G}_{R2} \mathbf{D}_{ii} & -\mathbf{G}_{R2} \mathbf{M}_{ii} \\ \mathbf{G}_{R2} \mathbf{K}_{ib} & \frac{1}{2} \mathbf{G}_{R2} \mathbf{D}_{ib} & \mathbf{G}_{R2} \mathbf{K}_{ii} & \mathbf{F}_{R2} + \frac{1}{2} \mathbf{G}_{R2} \mathbf{D}_{ii} \end{array} \right] \quad (9.96)$$

$$\underline{\mathbf{T}}_L^{-1} = \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{T}_{Lbi}^T & \mathbf{T}_{Lii} \end{bmatrix} = \left[\begin{array}{cc|cc} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \hline -\frac{1}{2} \mathbf{G}_{L2} \mathbf{D}_{bi}^T & -\mathbf{G}_{L2} \mathbf{M}_{bi}^T & \mathbf{F}_{L2} - \frac{1}{2} \mathbf{G}_{L2} \mathbf{D}_{ii}^T & -\mathbf{G}_{L2} \mathbf{M}_{ii}^T \\ \mathbf{G}_{L2} \mathbf{K}_{bi}^T & \frac{1}{2} \mathbf{G}_{L2} \mathbf{D}_{bi}^T & \mathbf{G}_{L2} \mathbf{K}_{ii}^T & \mathbf{F}_{L2} + \frac{1}{2} \mathbf{G}_{L2} \mathbf{D}_{ii}^T \end{array} \right] \quad (9.97)$$

The extended the right and left Craig Bampton transformations $\underline{\mathbf{E}}_R, \underline{\mathbf{E}}_L$ are defined in the forms

$$\underline{\mathbf{E}}_R = \underline{\mathbf{T}}_R, \quad \underline{\mathbf{E}}_L = \underline{\mathbf{T}}_L \quad (9.98)$$

where the right and left transformations $\underline{\mathbf{T}}_R, \underline{\mathbf{T}}_L$ are the inverse of equations (9.96), (9.97) and can be written as

$$\underline{\mathbf{T}}_R = \begin{bmatrix} \mathbf{I} & 0 \\ -(\mathbf{T}_{Rii}^{-1} \mathbf{T}_{Rib}) & \mathbf{T}_{Rii}^{-1} \end{bmatrix}, \quad \underline{\mathbf{T}}_L = \begin{bmatrix} \mathbf{I} & 0 \\ -(\mathbf{T}_{Lbi}^T \mathbf{T}_{Lii}^{-1}) & \mathbf{T}_{Lii}^{-1} \end{bmatrix} \quad (9.99)$$

9.5.3 Craig-Bampton transforms $\underline{\mathbf{F}}_R, \underline{\mathbf{F}}_L$ in the full size of the substructure

The first part of the Craig Bampton transformation matrices $\mathbf{F}_R, \mathbf{F}_L$ represent Guyan reduction transformations, where

$$\mathbf{F}_R = -\mathbf{K}_{ii}^{-1} \mathbf{K}_{ib} \quad , \quad \mathbf{F}_L = -\mathbf{K}_{ii}^{-T} \mathbf{K}_{bi}^T \quad (9.100)$$

The Craig Bampton transformations $\underline{\mathbf{F}}_R, \underline{\mathbf{F}}_L$ are extended to the full size of the substructure.

$$\underline{\mathbf{F}}_R = \begin{bmatrix} \mathbf{I} \\ \mathbf{F}_R \end{bmatrix} \quad , \quad \underline{\mathbf{F}}_L = \begin{bmatrix} \mathbf{I} \\ \mathbf{F}_L \end{bmatrix} \quad (9.101)$$

9.5.3.1 Extending $\underline{\mathbf{F}}_R, \underline{\mathbf{F}}_L$ to the full size of the substructure

Garvey *et al.* [43] proposed a logical extension of the Guyan reduction method for general damped systems. Following the proposed approach for structure preserving transformations, Craig Bampton transformations $\underline{\mathbf{F}}_R, \underline{\mathbf{F}}_L$ can be expressed in two parts as follows

$$\underline{\mathbf{F}}_R = \underline{\mathbf{U}}_{R1} \underline{\mathbf{U}}_{R2} \quad , \quad \underline{\mathbf{F}}_L = \underline{\mathbf{U}}_{L1} \underline{\mathbf{U}}_{L2} \quad (9.102)$$

The first transformations $\{\underline{\mathbf{U}}_{R1}, \underline{\mathbf{U}}_{L1}\}$ are square matrices and can be expressed in the form

$$\underline{\mathbf{U}}_{R1} = \begin{bmatrix} \mathbf{I} & 0 & 0 & 0 \\ 0 & \mathbf{I} & 0 & 0 \\ \mathbf{W}_{FR} & \mathbf{X}_{FR} & \mathbf{I} & 0 \\ \mathbf{Y}_{FR} & \mathbf{Z}_{FR} & 0 & \mathbf{I} \end{bmatrix} \quad (9.103)$$

$$\underline{\mathbf{U}}_{L1} = \begin{bmatrix} \mathbf{I} & 0 & 0 & 0 \\ 0 & \mathbf{I} & 0 & 0 \\ \mathbf{W}_{FL} & \mathbf{X}_{FL} & \mathbf{I} & 0 \\ \mathbf{Y}_{FL} & \mathbf{Z}_{FL} & 0 & \mathbf{I} \end{bmatrix} \quad (9.104)$$

more concisely

$$\underline{\mathbf{U}}_{R1} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{U}_{R1} & \mathbf{I} \end{bmatrix}, \quad \underline{\mathbf{U}}_{L1} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{U}_{L1} & \mathbf{I} \end{bmatrix} \quad (9.105)$$

The second transformations $\{\underline{\mathbf{U}}_{R2}, \underline{\mathbf{U}}_{L2}\}$ can be written as

$$\underline{\mathbf{U}}_{R2} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \underline{\mathbf{U}}_{L2} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (9.106)$$

The extended Craig Bampton transformations $\underline{\mathbf{F}}_R, \underline{\mathbf{F}}_L$ in the full size of the substructure are defined as

$$\underline{\mathbf{F}}_R = \underline{\mathbf{U}}_{R1}, \quad \underline{\mathbf{F}}_L = \underline{\mathbf{U}}_{L1} \quad (9.107)$$

Then, Craig Bampton transformations $\underline{\mathbf{T}}_{Rcb}, \underline{\mathbf{T}}_{Lcb}$ in SPEs form are defined from the multiplication of equation (9.103) by the inverse of (9.96) and equation (9.104) by the inverse of equation (9.97).

$$\underline{\mathbf{T}}_{Rcb} = \underline{\mathbf{U}}_{R1} \underline{\mathbf{E}}_R, \quad \underline{\mathbf{T}}_{Lcb} = \underline{\mathbf{U}}_{L1} \underline{\mathbf{E}}_L \quad (9.108)$$

$$\underline{\mathbf{T}}_{Rcb} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{U}_{R1} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -(\mathbf{T}_{Rii}^{-1} \mathbf{T}_{Rib}) & \mathbf{T}_{Rii}^{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{U}_{R1} - (\mathbf{T}_{Rii}^{-1} \mathbf{T}_{Rib}) & \mathbf{T}_{Rii}^{-1} \end{bmatrix} \quad (9.109)$$

$$\underline{\mathbf{T}}_{Lcb} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{U}_{L1} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -(\mathbf{T}_{Lbi}^T \mathbf{T}_{Lii}^{-1}) & \mathbf{T}_{Lii}^{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{U}_{L1} - (\mathbf{T}_{Lbi}^T \mathbf{T}_{Lii}^{-1}) & \mathbf{T}_{Lii}^{-1} \end{bmatrix} \quad (9.110)$$

9.6 Modified Craig Bampton transformations and SPEs

Craig Bampton transformations for general damped substructure systems are modified such that

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K}_s \\ \mathbf{K}_s & \mathbf{D}_s \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_A \\ \mathbf{K}_A & \mathbf{D}_A \end{bmatrix} \quad (9.111)$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{K}_s & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_s \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{K}_A & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_A \end{bmatrix} \quad (9.112)$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} -\mathbf{D}_s & -\mathbf{M}_s \\ -\mathbf{M}_s & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} -\mathbf{D}_A & -\mathbf{M}_A \\ -\mathbf{M}_A & \mathbf{0} \end{bmatrix} \quad (9.113)$$

The modified Craig Bampton transformations matrices are chosen in form to satisfy specific structure. The matrices $\{\mathbf{M}_A, \mathbf{D}_A, \mathbf{K}_A\}$ are chosen to be in the form

$$\mathbf{M}_A = \begin{bmatrix} \mathbf{M}_{Abb} & \mathbf{M}_{Abi} \\ \mathbf{M}_{Aib} & \mathbf{M}_{Aii} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{Abb} & \mathbf{M}_{Abi} \\ \mathbf{M}_{Aib} & \mathbf{M}_{Aii} \end{bmatrix} \quad (9.114)$$

$$\mathbf{D}_A = \begin{bmatrix} \mathbf{D}_{Abb} & \mathbf{D}_{Abi} \\ \mathbf{D}_{Aib} & \mathbf{D}_{Aii} \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{Abb} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{Aii} \end{bmatrix} \quad (9.115)$$

$$\mathbf{K}_A = \begin{bmatrix} \mathbf{K}_{Abb} & \mathbf{K}_{Abi} \\ \mathbf{K}_{Aib} & \mathbf{K}_{Aii} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{Abb} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{Aii} \end{bmatrix} \quad (9.116)$$

Matrices $\{\mathbf{M}_{Aii}, \mathbf{D}_{Aii}, \mathbf{K}_{Aii}\}$ are all diagonal and $\mathbf{K}_{Abi} = \mathbf{0} = \mathbf{D}_{Abi}$ and $\mathbf{K}_{Aib} = \mathbf{0} = \mathbf{D}_{Aib}$. The right and left modified Craig Bampton transformations $\{\underline{\mathbf{T}}_{Rcb}, \underline{\mathbf{T}}_{Lcb}\}$ have two parts as in equation (9.67). The two parts of the modified Craig Bampton transformations $\{\underline{\mathbf{E}}_R, \underline{\mathbf{E}}_L\}$ and $\{\underline{\mathbf{F}}_R, \underline{\mathbf{F}}_L\}$ are modified separately.

9.6.1 Modified Craig-Bampton transforms $\{\underline{\mathbf{E}}_R, \underline{\mathbf{E}}_L\}$ in the full size of the substructure

The matrices $\mathbf{G}_{R2_}, \mathbf{G}_{L2_}$ and $\mathbf{F}_{R2_}, \mathbf{F}_{L2_}$ in equations (9.94) and (9.95) have dimensions $(n \times n)$, and are formed as

$$\mathbf{G}_{R2_} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{G}_{R2} \end{bmatrix}, \quad \mathbf{G}_{L2_} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{G}_{L2} \end{bmatrix} \quad (9.117)$$

$$\mathbf{F}_{R2_} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{F}_{R2} \end{bmatrix}, \quad \mathbf{F}_{L2_} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{F}_{L2} \end{bmatrix} \quad (9.118)$$

The right and left transformations can be written as

$$\underline{\mathbf{T}}_R^{-1} = \left[\begin{array}{cc|cc} \mathbf{I} & 0 & 0 & 0 \\ -\frac{1}{2}\mathbf{G}_{R2}\mathbf{D}_{ib} & \mathbf{F}_{R2} - \frac{1}{2}\mathbf{G}_{R2}\mathbf{D}_{ii} & -\mathbf{G}_{R2}\mathbf{M}_{ib} & -\mathbf{G}_{R2}\mathbf{M}_{ii} \\ \hline 0 & 0 & \mathbf{I} & 0 \\ \mathbf{G}_{R2}\mathbf{K}_{ib} & \mathbf{G}_{R2}\mathbf{K}_{ii} & \frac{1}{2}\mathbf{G}_{R2}\mathbf{D}_{ib} & \mathbf{F}_{R2} + \frac{1}{2}\mathbf{G}_{R2}\mathbf{D}_{ii} \end{array} \right] \quad (9.119)$$

$$\underline{\mathbf{T}}_L^{-1} = \left[\begin{array}{cc|cc} \mathbf{I} & 0 & 0 & 0 \\ -\frac{1}{2}\mathbf{G}_{L2}\mathbf{D}_{bi}^T & \mathbf{F}_{L2} - \frac{1}{2}\mathbf{G}_{L2}\mathbf{D}_{ii}^T & -\mathbf{G}_{L2}\mathbf{M}_{bi}^T & -\mathbf{G}_{L2}\mathbf{M}_{ii}^T \\ \hline 0 & 0 & \mathbf{I} & 0 \\ \mathbf{G}_{L2}\mathbf{K}_{bi}^T & \mathbf{G}_{L2}\mathbf{K}_{ii}^T & \frac{1}{2}\mathbf{G}_{L2}\mathbf{D}_{bi}^T & \mathbf{F}_{L2} + \frac{1}{2}\mathbf{G}_{L2}\mathbf{D}_{ii}^T \end{array} \right] \quad (9.120)$$

The modified right and left Craig Bampton transformations $\underline{\mathbf{E}}_R, \underline{\mathbf{E}}_L$ in the full size of the substructure are defined as

$$\underline{\mathbf{E}}_R = \left(\underline{\mathbf{T}}_R^{-1} \right)^{-1}, \quad \underline{\mathbf{E}}_L = \left(\underline{\mathbf{T}}_L^{-1} \right)^{-1} \quad (9.121)$$

9.6.2 *Modified Craig-Bampton transforms $\underline{\mathbf{F}}_R, \underline{\mathbf{F}}_L$ in the full size of the substructure*

The extended Craig Bampton transformations $\underline{\mathbf{F}}_R, \underline{\mathbf{F}}_L$ in equations (9.103) and (9.104) are rearranged to form

$$\underline{\mathbf{U}}_{R1} = \begin{bmatrix} \mathbf{I} & 0 & 0 & 0 \\ \mathbf{W}_{FR} & \mathbf{I} & \mathbf{X}_{FR} & 0 \\ 0 & 0 & \mathbf{I} & 0 \\ \mathbf{Y}_{FR} & 0 & \mathbf{Z}_{FR} & \mathbf{I} \end{bmatrix} \quad (9.122)$$

$$\underline{\mathbf{U}}_{L1} = \begin{bmatrix} \mathbf{I} & 0 & 0 & 0 \\ \mathbf{W}_{FL} & \mathbf{I} & \mathbf{X}_{FL} & 0 \\ 0 & 0 & \mathbf{I} & 0 \\ \mathbf{Y}_{FL} & 0 & \mathbf{Z}_{FL} & \mathbf{I} \end{bmatrix} \quad (9.123)$$

The right and left modified Craig Bampton transformations $\underline{\mathbf{F}}_R, \underline{\mathbf{F}}_L$ in the full size of the substructure are defined as

$$\underline{\mathbf{F}}_R = \underline{\mathbf{U}}_{R1} = \begin{bmatrix} \mathbf{I} & 0 & 0 & 0 \\ \mathbf{W}_{FR} & \mathbf{I} & \mathbf{X}_{FR} & 0 \\ 0 & 0 & \mathbf{I} & 0 \\ \mathbf{Y}_{FR} & 0 & \mathbf{Z}_{FR} & \mathbf{I} \end{bmatrix} \quad (9.124)$$

$$\underline{\mathbf{F}}_L = \underline{\mathbf{U}}_{L1} = \begin{bmatrix} \mathbf{I} & 0 & 0 & 0 \\ \mathbf{W}_{FL} & \mathbf{I} & \mathbf{X}_{FL} & 0 \\ 0 & 0 & \mathbf{I} & 0 \\ \mathbf{Y}_{FL} & 0 & \mathbf{Z}_{FL} & \mathbf{I} \end{bmatrix} \quad (9.125)$$

The Craig Bampton transformations $\{\underline{\mathbf{T}}_{Rcb}, \underline{\mathbf{T}}_{Lcb}\}$ for general damped systems are modified to preserve *LAMs* matrices in new forms as equations (9.114), (9.115) and (9.116). The modified Craig Bampton transformations $\{\underline{\mathbf{T}}_{Rcb}, \underline{\mathbf{T}}_{Lcb}\}$ are defined from multiplication of equation (9.122) by inverse of equation (9.119) and equation (9.123) by inverse of equation (9.120).

$$\underline{\mathbf{T}}_{Rcb} = \underline{\mathbf{F}}_R \underline{\mathbf{E}}_R \quad , \quad \underline{\mathbf{T}}_{Lcb} = \underline{\mathbf{F}}_L \underline{\mathbf{E}}_L \quad (9.126)$$

$$\underline{\mathbf{T}}_{Rcb} = \begin{bmatrix} \mathbf{I} & 0 & 0 & 0 \\ \mathbf{W}_{FR} & \mathbf{I} & \mathbf{X}_{FR} & 0 \\ 0 & 0 & \mathbf{I} & 0 \\ \mathbf{Y}_{FR} & 0 & \mathbf{Z}_{FR} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 & 0 & 0 \\ -\frac{1}{2}\mathbf{G}_{R2}\mathbf{D}_{ib} & \mathbf{F}_{R2} - \frac{1}{2}\mathbf{G}_{R2}\mathbf{D}_{ii} & -\mathbf{G}_{R2}\mathbf{M}_{ib} & -\mathbf{G}_{R2}\mathbf{M}_{ii} \\ 0 & 0 & \mathbf{I} & 0 \\ \mathbf{G}_{R2}\mathbf{K}_{ib} & \mathbf{G}_{R2}\mathbf{K}_{ii} & \frac{1}{2}\mathbf{G}_{R2}\mathbf{D}_{ib} & \mathbf{F}_{R2} + \frac{1}{2}\mathbf{G}_{R2}\mathbf{D}_{ii} \end{bmatrix}^{-1} \quad (9.127)$$

$$\underline{\mathbf{T}}_{Lcb} = \begin{bmatrix} \mathbf{I} & 0 & 0 & 0 \\ \mathbf{W}_{FL} & \mathbf{I} & \mathbf{X}_{FL} & 0 \\ 0 & 0 & \mathbf{I} & 0 \\ \mathbf{Y}_{FL} & 0 & \mathbf{Z}_{FL} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 & 0 & 0 \\ -\frac{1}{2}\mathbf{G}_{L2}\mathbf{D}_{bi}^T & \mathbf{F}_{L2} - \frac{1}{2}\mathbf{G}_{L2}\mathbf{D}_{ii}^T & -\mathbf{G}_{L2}\mathbf{M}_{bi}^T & -\mathbf{G}_{L2}\mathbf{M}_{ii}^T \\ 0 & 0 & \mathbf{I} & 0 \\ \mathbf{G}_{L2}\mathbf{K}_{bi}^T & \mathbf{G}_{L2}\mathbf{K}_{ii}^T & \frac{1}{2}\mathbf{G}_{L2}\mathbf{D}_{bi}^T & \mathbf{F}_{L2} + \frac{1}{2}\mathbf{G}_{L2}\mathbf{D}_{ii}^T \end{bmatrix}^{-1} \quad (9.128)$$

9.7 Example

In this example, we consider a second order general linear damping subsystem. The motivation of this example is to calculate the Craig-Bampton transformations in state space form which are used as model reduction transformations. The subsystem matrices are non-symmetric with four degrees of freedom.

$$\mathbf{K}_s = \begin{bmatrix} -121 & 4715 & -1868 & -1662 \\ 3915 & -3289 & -367 & -2532 \\ 2623 & -3640 & -1606 & 4941 \\ 1553 & 2552 & -4904 & 4056 \end{bmatrix}, \quad \mathbf{D}_s = \begin{bmatrix} -15 & 40 & 24 & 36 \\ -22 & -18 & -39 & 32 \\ -41 & 7 & -12 & -25 \\ -26 & -31 & 36 & 40 \end{bmatrix},$$

$$\mathbf{M}_s = \begin{bmatrix} 104 & 479 & -50 & -3 \\ -360 & 101 & 143 & -425 \\ -339 & -138 & -255 & 267 \\ 383 & -269 & -32 & -455 \end{bmatrix}$$

The system matrices are expressed in state space form. Based on the division of the boundary coordinates and interior coordinates, the LAMs matrices are arranged as equations (9.42), (9.43), and (9.44). The number of boundary coordinates is chosen as two degrees of freedom. The right and left diagonalising SPEs for the interior eigenvalue problem are calculated

$$\underline{\mathbf{T}}_{Ri} = \begin{bmatrix} 0.0388 & -0.0378 & -0.0136 & -0.0137 \\ 0.0296 & -0.0295 & -0.0044 & -0.0041 \\ 0.1587 & 0.1665 & -0.0138 & 0.0135 \\ 0.0509 & 0.0494 & 0.0127 & -0.0143 \end{bmatrix}, \quad \underline{\mathbf{T}}_{Li} = \begin{bmatrix} 0.0694 & -0.0693 & -0.0076 & -0.0083 \\ -0.0010 & 0.0020 & -0.0103 & -0.0095 \\ 0.0892 & 0.1007 & 0.0398 & -0.0383 \\ 0.1204 & 0.1152 & -0.0409 & 0.0375 \end{bmatrix}$$

The first part of Craig-Bampton transformations $\{\underline{\mathbf{F}}_R, \underline{\mathbf{F}}_L\}$ in state space are calculated using equation (9.55) based on boundary coordinates.

$$\underline{\mathbf{F}}_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.1674 & 1.5451 & 0 & 0 \\ -0.5853 & 1.2389 & 0 & 0 \\ 0 & 0 & -0.4173 & -1.0806 \\ 0 & 0 & 0.8711 & -0.5152 \end{bmatrix}, \quad \underline{\mathbf{F}}_L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.8877 & 0.7849 & 0 & 0 \\ -0.6716 & -0.3319 & 0 & 0 \\ 0 & 0 & -0.1819 & 0.6315 \\ 0 & 0 & -0.1133 & -0.5635 \end{bmatrix}$$

The second part of Craig-Bampton transformations $\{\underline{\mathbf{E}}_R, \underline{\mathbf{E}}_L\}$ in state space are calculated from equation (9.70) in terms of diagonalising SPEs for the interior coordinates

$$\underline{\mathbf{E}}_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.0388 & -0.0378 & -0.0136 & -0.0137 \\ 0.0296 & -0.0295 & -0.0044 & -0.0041 \\ 0.1587 & 0.1665 & -0.0138 & 0.0135 \\ 0.0509 & 0.0494 & 0.0127 & -0.0143 \end{bmatrix}, \quad \underline{\mathbf{E}}_L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.0694 & -0.0693 & -0.0076 & -0.0083 \\ -0.0010 & 0.0020 & -0.0103 & -0.0095 \\ 0.0892 & 0.1007 & 0.0398 & -0.0383 \\ 0.1204 & 0.1152 & -0.0409 & 0.0375 \end{bmatrix}$$

The two parts of Craig-Bampton transformations $\{\underline{\mathbf{F}}_R, \underline{\mathbf{F}}_L, \underline{\mathbf{E}}_R, \underline{\mathbf{E}}_L\}$ satisfy equations (9.64), (9.65) and (9.66). The right and left Craig-Bampton transformations $\{\underline{\mathbf{T}}_{Rcb}, \underline{\mathbf{T}}_{Lcb}\}$ are calculated from equation (9.67) which satisfy equations (9.75), (9.76) and (9.77).

$$\underline{\mathbf{M}}_N = \left[\begin{array}{cccc|cccc} 0 & 0 & 1164.4 & -230.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5458.3 & -6992.9 & 0 & 0 & 0 & 0 \\ 1164.4 & -230.2 & 17.6 & -0.2 & -123.6 & 133.1 & 27.2 & 26.0 \\ 5458.3 & -6992.9 & -10.8 & 79.4 & -7.5 & -13.7 & -10.5 & -12.8 \\ \hline 0 & 0 & 514.1 & -322.3 & 0 & 0 & 11.7 & 0 \\ 0 & 0 & -518.8 & 313.1 & 0 & 0 & 0 & 12.2 \\ 0 & 0 & -133.5 & -20.4 & 11.7 & 0 & -3.9 & 0 \\ 0 & 0 & -128.4 & -22.4 & 0 & 12.2 & 0 & 3.7 \end{array} \right]$$

$$\underline{\mathbf{D}}_N = \left[\begin{array}{cccc|cccc} 1164.4 & -230.2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5458.3 & -6992.9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -122.3 & -534.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 789.9 & -165.4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 11.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{array} \right]$$

$$\underline{\mathbf{K}}_N = \left[\begin{array}{cccc|cccc} 24.5 & -34.1 & -122.3 & -534.6 & 13.8 & 14.9968 & -10.6 & 10.9 \\ 34.9 & 96.6 & 789.9 & -165.4 & 13.0 & 8.6 & -0.6 & -1.1 \\ -122.2 & -534.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 789.9 & -165.4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -36.8 & 127.2 & 0 & 0 & 3.9 & 0 & -1 & 0 \\ -31.4 & 118.8 & 0 & 0 & 0 & -3.7 & 0 & -1 \\ 44.01 & -27.6 & 0 & 0 & -1 & 0 & 0 & 0 \\ -42.6 & 25.7 & 0 & 0 & 0 & -1 & 0 & 0 \end{array} \right]$$

The reduce system matrices $\{\underline{\mathbf{M}}_{cb}, \underline{\mathbf{D}}_{cb}, \underline{\mathbf{K}}_{cb}\}$ are calculated from equations (9.81), (9.82) and (9.83).

$$\underline{\mathbf{M}}_{cb} = \begin{bmatrix} 0 & 0 & 1164.4 & -230.2 \\ 0 & 0 & 5458.3 & -6992.9 \\ 1164.4 & -230.2 & 17.6 & -0.2 \\ 5458.3 & -6992.9 & -10.8 & 79.4 \end{bmatrix}, \quad \underline{\mathbf{D}}_{cb} = \begin{bmatrix} 1164.4 & -230.2 & 0 & 0 \\ 5458.3 & -6992.9 & 0 & 0 \\ 0 & 0 & -122.3 & -534.6 \\ 0 & 0 & 789.9 & -165.4 \end{bmatrix}$$

$$\underline{\mathbf{K}}_{cb} = \begin{bmatrix} 24.5 & -34.1 & -122.3 & -534.6 \\ 34.9 & 96.6 & 789.9 & -165.4 \\ -122.3 & -534.6 & 0 & 0 \\ 789.9 & -165.4 & 0 & 0 \end{bmatrix}$$

The Craig Bampton transformations preserve the substructure system matrices in the same format as *LAMs* matrices structure.

9.8 Conclusion

This chapter has investigated Craig Bampton transforms $\{\underline{\mathbf{T}}_{Rcb}, \underline{\mathbf{T}}_{Lcb}\}$ for undamped systems. The ideas for undamped systems have been extended to the generally damped systems in terms of diagonalising SPEs. New expressions of Craig Bampton transformations are presented. These transformations have been developed to preserve the substructure matrices in the same format as *LAMs* matrices structure. The developed Craig Bampton transformations have been used as model reduction transformations. The two parts of Craig Bampton transformations are extended to the full size of the substructure. The first part of the Craig Bampton transformations is extracted from a logical extension of Garvey *et al.* [43] to the full size of the substructure. The second part of the Craig Bampton transformations represents diagonalising SPEs in the full size of the substructure which diagonalise the interior coordinates.

CHAPTER 10. Conclusions and further work

10.1 Conclusions

The ambition of this study was divided into three aspects. (1) Developing and generalising modal correlation methods for undamped systems and extending these ideas to general damped systems. (2) Comparing the newly developed methods with alternative methods. (3) Applying these methods to model updating for general damped systems.

This thesis starts with modelling an aircraft (GARTEUR-like) structure with a 3D finite element model. Modal parameters (the natural frequencies and mode shapes) are calculated using the system models. These procedures are described in Chapter 3. A general 3D finite element code has been developed in MATLABTM for this purpose.

Experimental modal analysis was carried out on a physical implementation of the GARTEUR structure. This was used to extract modal data which were identified from a single column of frequency response. The modal data represent natural frequency and the modal damping ratio for each single mode of the vibrating system. The modal analysis procedures are described in Chapter 4.

This thesis has shown how existing methods for eigenvalue and eigenvalue derivatives for undamped systems can be extended to the concept of structure-preserving equivalences to yield general methods for calculating the derivatives of

both the diagonalised system and the diagonalising transformations. This approach resolves the quandary where eigenvalue and eigenvector derivatives become undefined when a pair of complex eigenvalues turns into a pair of real eigenvalues or vice-versa. This has also resolved the problem encountered when any one or more of the system matrices is singular. These methods are described in Chapter 5, are needed for model updating.

New modal correlation methods have been developed and implemented initially for undamped systems in all cases. These ideas have been subsequently extended for general damped linear systems. *Cross orthogonality* measures and *mutual orthogonality* measures were investigated. These measures were defined in terms of real matrices - the *diagonalising structure preserving equivalences* (DSPEs).

Cross orthogonality measures require the right and left modal information for the physical system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ and analytical system $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$. The modes of the analytical system must be correctly matched to the modes of the measured system. *Mutual orthogonality* measures do not require any modal information from the analytical system $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$. This matching of modes is unnecessary. However, for systems having non-symmetric matrices, both right and left modes are required for the *mutual orthogonality* measures.

The presented methods have been well developed for ill-conditioned systems such that they work for all occasions and not only for cases where mass matrix is non-singular. A measure of the residuals was introduced which did not demand invertibility of diagonalised system matrices. This procedure is introduced in Chapter

7. Möbius/spectral transformation formulae were used to address a problem where the mass matrix is singular. The Möbius transformation is decomposed into four elementary spectral transformations. These spectral transformations are used to find a relationship between the diagonalising SPEs for the new system matrices and the original system matrices. Two new transformations are investigated called system spectral transformation \mathbf{Q}_{SSTN} and diagonalising spectral/similarity transformation \mathbf{Q}_{DSTO} . The transformation \mathbf{Q}_{SSTN} maps between two systems having the same short eigenvectors and their diagonalised system matrices. The transformation \mathbf{Q}_{DSTO} maps between two SPEs having identical eigenvalues. This procedure is described in Chapter 6.

The newly developed modal correlation methods have been used for model updating. The presented model updating methods in this thesis have been developed for undamped systems and extended to general damped systems. Alternative methods were compared with the developed modal correlation methods for model updating. The model updating methods have been implemented based on eigenvectors and SPEs and both *cross-orthogonality* measures and *mutual-orthogonality* measures. In both *cross-orthogonality* and *mutual-orthogonality* measures cases, these measures were defined in terms of real matrices - the *diagonalising structure preserving equivalences* (DSPEs).

The model updating methods based on *cross orthogonality* measures require the right and left modal information for the physical system $\{\mathbf{K}_o, \mathbf{D}_o, \mathbf{M}_o\}$ and analytical

system $\{\mathbf{K}_a, \mathbf{D}_a, \mathbf{M}_a\}$. The modes of the measured system must be correctly matched to the modes of the true system.

The model updating method based on the *mutual orthogonality* measures exhibits monotonic convergence from every starting position. The ball of convergence has an infinite radius. This contrasts starkly with updating procedures based on comparing eigenvectors which exhibit a finite ball of convergence.

Craig Bampton transformations were presented for undamped systems. The developed Craig Bampton transformations have been extended for damped second order systems in state space. New expressions of Craig Bampton transformations have been presented in SPEs forms. The two parts of the Craig Bampton transformations were extended in the full sizes of the substructure. The extended Craig Bampton transformations have been modified to format each block of transformed substructure matrices as *LAMs* matrices format. These procedures are described in Chapter 9.

10.2 Further work

The presented study presents some interesting possibilities for further investigations. Some recommendations for further work are presented here

10.2.1 Applications on real aircraft structures

The developed methods in this study have been applied on deliberately chosen examples. These methods can be applied to real aircraft structure with real structural damage. These methods can also be implemented as a part of airworthiness maintenance schedule programs for large complex aircraft structures.

10.2.2 Excitation and response devices

The modal testing carried out in this project has used contact excitation achieved with an impact hammer and response measured with a piezoelectric accelerometer. The type of analysis to be performed and accuracy requirements of test information influence the choice of excitation signal. Non contact devices of input and output signals can be applied instead of the contact devices. Thermal excitation is a non contact exciter which can be used to excite aircraft models.

APPENDICES

Appendix A. Proof of diagonalising SPEs $\{\underline{\mathbf{T}}_R, \underline{\mathbf{T}}_L\}$

The Diagonalising SPEs matrices $\{\underline{\mathbf{T}}_L, \underline{\mathbf{T}}_R\}$ diagonalise LAMs matrices $\{\underline{\mathbf{K}}, \underline{\mathbf{D}}, \underline{\mathbf{M}}\}$ for the system $\{\underline{\mathbf{K}}, \underline{\mathbf{D}}, \underline{\mathbf{M}}\}$ such that

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_D \\ \mathbf{K}_D & \mathbf{D}_D \end{bmatrix} \quad (\text{A.1})$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{K}_D & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_D \end{bmatrix} \quad (\text{A.2})$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} -\mathbf{D}_D & -\mathbf{M}_D \\ -\mathbf{M}_D & \mathbf{0} \end{bmatrix} \quad (\text{A.3})$$

where $\{\underline{\mathbf{K}}_D, \underline{\mathbf{D}}_D, \underline{\mathbf{M}}_D\}$ are block diagonal matrices. The new system matrices $\{\underline{\mathbf{K}}_D, \underline{\mathbf{D}}_D, \underline{\mathbf{M}}_D\}$ represent the LAMs for the system whose coefficient matrices are the diagonal matrices $\{\underline{\mathbf{K}}_D, \underline{\mathbf{D}}_D, \underline{\mathbf{M}}_D\}$. These transformations $\{\underline{\mathbf{T}}_L, \underline{\mathbf{T}}_R\}$ can be expressed as

$$\underline{\mathbf{T}}_L = \begin{bmatrix} \mathbf{F}_L - \frac{1}{2} \mathbf{G}_L \mathbf{D}_D & -\mathbf{G}_L \mathbf{M}_D \\ (\mathbf{G}_L \mathbf{K}_D) & \mathbf{F}_L + \frac{1}{2} \mathbf{G}_L \mathbf{D}_D \end{bmatrix} \quad (\text{A.4})$$

$$\underline{\mathbf{T}}_R = \begin{bmatrix} \mathbf{F}_R - \frac{1}{2} \mathbf{G}_R \mathbf{D}_D & -\mathbf{G}_R \mathbf{M}_D \\ \mathbf{G}_R \mathbf{K}_D & \mathbf{F}_R + \frac{1}{2} \mathbf{G}_R \mathbf{D}_D \end{bmatrix} \quad (\text{A.5})$$

where $\{\mathbf{F}_L, \mathbf{F}_R, \mathbf{G}_L, \mathbf{G}_R\}$ are real-valued $(n \times n)$ matrices satisfying the constraint:

$$\mathbf{F}_R \mathbf{G}_L^T + \mathbf{G}_R \mathbf{F}_L^T = \mathbf{0} \quad (\text{A.6})$$

It is possible to prove equations (2.79) and (2.80) by inverting equation (2.73)

$$\begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} & -\mathbf{M}^{-1} \\ -\mathbf{M}^{-1} & \mathbf{M}^{-1} \mathbf{D} \mathbf{M}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^{-T} = \begin{bmatrix} \mathbf{0} & -\mathbf{M}_D^{-1} \\ -\mathbf{M}_D^{-1} & \mathbf{M}_D^{-1} \mathbf{D}_D \mathbf{M}_D^{-1} \end{bmatrix} \quad (\text{A.7})$$

and pre-multiplying equation (2.72) by equation (A.7) to yield

$$\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}_D^{-1}\mathbf{K}_D & -\mathbf{M}_D^{-1}\mathbf{D}_D \end{bmatrix} \quad (\text{A.8})$$

There are four block equations with four unknowns. Four $(n \times n)$ square matrices $\{\mathbf{W}_R, \mathbf{X}_R, \mathbf{Y}_R, \mathbf{Z}_R\}$ can be calculated from equation (A.8). Using the top two block equations in equation (A.8)

$$\mathbf{Y}_R = -\mathbf{X}_R \mathbf{M}_D^{-1} \mathbf{K}_D \quad (\text{A.9})$$

$$\mathbf{Z}_R = \mathbf{W}_R - \mathbf{X}_R \mathbf{M}_D^{-1} \mathbf{D}_D \quad (\text{A.10})$$

Provided that \mathbf{M}_D is an invertible matrix and that for some \mathbf{G}_R

$$\mathbf{X}_R = -\mathbf{G}_R \mathbf{M}_D \quad (\text{A.11})$$

Then equations (A.9) and (A.10) become

$$\mathbf{Y}_R = \mathbf{G}_R \mathbf{K}_D \quad (\text{A.12})$$

$$\mathbf{Z}_R - \mathbf{W}_R = \mathbf{G}_R \mathbf{D}_D \quad (\text{A.13})$$

For some \mathbf{F}_R matrix let

$$\mathbf{Z}_R + \mathbf{W}_R = 2\mathbf{F}_R \quad (\text{A.14})$$

Using equations (A.13) and (A.14) yields

$$\mathbf{W}_R = \mathbf{F}_R - \frac{1}{2} \mathbf{G}_R \mathbf{D}_D \quad (\text{A.15})$$

$$\mathbf{Z}_R = \mathbf{F}_R + \frac{1}{2} \mathbf{G}_R \mathbf{D}_D \quad (\text{A.16})$$

This proves the form of equation (2.79).

Similarly, the left transformation can be calculated by post-multiplying equation (2.72) by equation (A.7)

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & -\mathbf{K} \mathbf{M}^{-1} \\ \mathbf{I} & -\mathbf{D} \mathbf{M}^{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{K}_D \mathbf{M}_D^{-1} \\ \mathbf{I} & -\mathbf{D}_D \mathbf{M}_D^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \quad (\text{A.17})$$

Taking the transpose of equation (A.17)

$$\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-T}\mathbf{K}^T & -\mathbf{M}^{-T}\mathbf{D}^T \end{bmatrix} \begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix} = \begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}_D^{-T}\mathbf{K}_D^T & -\mathbf{M}_D^{-T}\mathbf{D}_D^T \end{bmatrix} \quad (\text{A.18})$$

From the two block equations of (A.18)

$$\mathbf{Y}_L = -\mathbf{X}_L \mathbf{M}_D^{-T} \mathbf{K}_D^T \quad (\text{A.19})$$

$$\mathbf{Z}_L = \mathbf{W}_L - \mathbf{X}_L \mathbf{M}_D^{-T} \mathbf{D}_D^T \quad (\text{A.20})$$

Again suppose that for some \mathbf{G}_L

$$\mathbf{X}_L = -\mathbf{G}_L \mathbf{M}_D^T \quad (\text{A.21})$$

Matrices $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$ are diagonal; $\mathbf{M}_D = \mathbf{M}_D^T$, $\mathbf{D}_D = \mathbf{D}_D^T$, $\mathbf{K}_D = \mathbf{K}_D^T$. Equations

(A.19) and (A.20) can be written as

$$\mathbf{Y}_L = \mathbf{G}_L \mathbf{K}_D \quad (\text{A.22})$$

$$\mathbf{Z}_L - \mathbf{W}_L = \mathbf{G}_L \mathbf{D}_D \quad (\text{A.23})$$

For some \mathbf{F}_L matrix let

$$\mathbf{Z}_L + \mathbf{W}_L = 2\mathbf{F}_L \quad (\text{A.24})$$

Using equations (A.23) and (A.24) yields

$$\mathbf{W}_L = \mathbf{F}_L - \frac{1}{2} \mathbf{G}_L \mathbf{D}_D \quad (\text{A.25})$$

$$\mathbf{Z}_L = \mathbf{F}_L + \frac{1}{2} \mathbf{G}_L \mathbf{D}_D \quad (\text{A.26})$$

Appendix B. Binomial theorem

The binomial theorem formula for negative integer can be written as

$$(1-x)^{-r} = \sum_{k=0}^{\infty} \binom{r}{k} x^k = \sum_{k=0}^{\infty} \frac{(r)_k}{k!} x^k \quad (\text{B.1})$$

$$= 1 + rx + \frac{1}{2!} r(r+1)x^2 + \frac{1}{3!} r(r+1)(r+2)x^3 + \dots$$

where r is any real number and $k!$ is a factorial. The pochhammer symbol $(r)_k$ is written as

$$\binom{r}{k} = \frac{(r)_k}{k!} = \frac{r(r+1)\dots(r+k-1)}{k!} \quad (\text{B.2})$$

The model reduction transformation matrix is written in form

$$\mathbf{t}_d = (\mathbf{K}_{ee} - \omega^2 \mathbf{M}_{ee})^{-1} (\mathbf{K}_{er} - \omega^2 \mathbf{M}_{er}) \quad (\text{B.3})$$

Rearrange equation (B.3) and using the binomial series, the term $(\mathbf{K}_{ee} - \omega^2 \mathbf{M}_{ee})^{-1}$ is expanded to $\mathbf{K}_{ee}^{-1} (1 + \omega^2 \mathbf{K}_{ee}^{-1} \mathbf{M}_{ee})$ in which

$$(\mathbf{K}_{ee} - \omega^2 \mathbf{M}_{ee})^{-1} = \mathbf{K}_{ee}^{-1} (1 - \omega^2 \mathbf{M}_{ee} \mathbf{K}_{ee}^{-1})^{-1} \quad (\text{B.4})$$

and

$$(1 - \omega^2 \mathbf{M}_{ee} \mathbf{K}_{ee}^{-1})^{-1} = 1 + \omega^2 \mathbf{M}_{ee} \mathbf{K}_{ee}^{-1} + (\omega^2 \mathbf{M}_{ee} \mathbf{K}_{ee}^{-1})^2 \quad (\text{B.5})$$

The terms associated with ω^4 are neglected compared with terms associated with ω^2 .

Substitute equations (B.4) and (B.5) into equation (B.3), the transformation matrix \mathbf{t}_d can be written as

$$\mathbf{t}_d = -\mathbf{K}_{ee}^{-1} (\mathbf{K}_{er} + \omega^2 (\mathbf{M}_{ee} \mathbf{K}_{ee}^{-1} \mathbf{K}_{er} - \mathbf{M}_{er})) \quad (\text{B.6})$$

Appendix C. Proof of two equivalent statements

For any three scalars $\{k, d, m\}$ the following two statements are equivalent

$$\text{a) } \left[k \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} + d \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} + m \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \right] \begin{bmatrix} \mathbf{v}_i \\ \mathbf{w}_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (\text{C.1})$$

where $\{\mathbf{v}_i, \mathbf{w}_i\}$ are some $(n \times 2)$ matrices, with $\begin{bmatrix} \mathbf{v}_i \\ \mathbf{w}_i \end{bmatrix}$ having rank 2.

$$\text{b) } (\mathbf{K} + \lambda_i \mathbf{D} + \lambda_i^2 \mathbf{M}) \mathbf{v}_i = 0 \quad \text{for } i = 1, 2 \quad (\text{C.2})$$

where $\{\mathbf{v}_1, \mathbf{v}_2\}$ are some non-zero n -entries vectors and $\{\lambda_1, \lambda_2\}$ satisfy the following equation

$$(k + \lambda_i d + \lambda_i^2 m) = 0 \quad (\text{C.3})$$

This appendix shows a proof of these statements which are equivalent. Rearrange equation (C.1) which can be written as

$$\begin{bmatrix} (d\mathbf{K} - k\mathbf{D}) & (m\mathbf{K} - k\mathbf{M}) \\ (m\mathbf{K} - k\mathbf{M}) & (m\mathbf{D} - d\mathbf{M}) \end{bmatrix} \begin{bmatrix} \mathbf{v}_i \\ \mathbf{w}_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (\text{C.4})$$

we get these two equations

$$(d\mathbf{K} - k\mathbf{D}) \mathbf{v}_i + (m\mathbf{K} - k\mathbf{M}) \mathbf{w}_i = 0 \quad (\text{C.5})$$

$$(m\mathbf{K} - k\mathbf{M}) \mathbf{v}_i + (m\mathbf{D} - d\mathbf{M}) \mathbf{w}_i = 0 \quad (\text{C.6})$$

Multiplying equation (C.5) by λ_i / k and equation (C.6) by $1/m$ yields

$$\left(\lambda_i \frac{d}{k} \mathbf{K} - \lambda_i \mathbf{D} \right) \mathbf{v}_i + \left(\lambda_i \frac{m}{k} \mathbf{K} - \lambda_i \mathbf{M} \right) \mathbf{w}_i = 0 \quad (\text{C.7})$$

$$\left(\mathbf{K} - \frac{k}{m} \mathbf{M} \right) \mathbf{v}_i + \left(\mathbf{D} - \frac{d}{m} \mathbf{M} \right) \mathbf{w}_i = 0 \quad (\text{C.8})$$

Taking a linear combination of equations (C.7), (C.8) and using equation (C.3) yields

$$\begin{aligned}
& \left(\lambda_i \frac{d}{k} + 1 \right) \mathbf{K} \mathbf{v}_i - \lambda_i \mathbf{D} \mathbf{v}_i - \frac{k}{m} \mathbf{M} \mathbf{v}_i + \lambda_i \frac{m}{k} \mathbf{K} \mathbf{w}_i + \mathbf{D} \mathbf{w}_i - \left(\frac{d}{m} + \lambda_i \right) \mathbf{M} \mathbf{w}_i = 0 \\
& -\lambda_i^2 \frac{m}{k} \mathbf{K} \mathbf{v}_i - \lambda_i \mathbf{D} \mathbf{v}_i - \frac{k}{m} \mathbf{M} \mathbf{v}_i + \lambda_i \frac{m}{k} \mathbf{K} \mathbf{w}_i + \mathbf{D} \mathbf{w}_i + \frac{1}{\lambda_i} \frac{k}{m} \mathbf{M} \mathbf{w}_i = 0 \\
& -\lambda_i \left(\lambda_i \frac{m}{k} \mathbf{K} + \mathbf{D} + \frac{1}{\lambda_i} \frac{k}{m} \mathbf{M} \right) \mathbf{v}_i + \left(\lambda_i \frac{m}{k} \mathbf{K} + \mathbf{D} + \frac{1}{\lambda_i} \frac{k}{m} \mathbf{M} \right) \mathbf{w}_i = 0
\end{aligned} \tag{C.9}$$

If the system matrices $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ are non-singular then equation (C.9) becomes

$$-\lambda_i \mathbf{A} \mathbf{v}_i + \mathbf{A} \mathbf{w}_i = 0 \quad \Rightarrow \quad \lambda_i \mathbf{v}_i = \mathbf{w}_i \tag{C.10}$$

where $\mathbf{A} =: \left(\lambda_i \frac{m}{k} \mathbf{K} + \mathbf{D} + \frac{1}{\lambda_i} \frac{k}{m} \mathbf{M} \right)$

Substituting equation (C.10) into equations (C.5), (C.6) and multiplying by $(-\lambda_i / m)$, $(1/m)$ respectively and using equation (C.3) yields

$$\begin{aligned}
& -\lambda_i \left(\frac{d}{m} \mathbf{K} - \frac{k}{m} \mathbf{D} \right) \mathbf{v}_i - \lambda_i^2 \left(\mathbf{K} - \frac{k}{m} \mathbf{M} \right) \mathbf{v}_i = \left(- \left(\lambda_i \frac{d}{m} + \lambda_i^2 \right) \mathbf{K} + \frac{k}{m} \lambda_i \mathbf{D} + \frac{k}{m} \lambda_i^2 \mathbf{M} \right) \mathbf{v}_i = 0 \\
& \left(\frac{k}{m} \mathbf{K} + \frac{k}{m} \lambda_i \mathbf{D} + \frac{k}{m} \lambda_i^2 \mathbf{M} \right) \mathbf{v}_i = \left(\mathbf{K} + \lambda_i \mathbf{D} + \lambda_i^2 \mathbf{M} \right) \mathbf{v}_i = 0
\end{aligned} \tag{C.11}$$

$$\begin{aligned}
& \left(\mathbf{K} - \frac{k}{m} \mathbf{M} \right) \mathbf{v}_i + \lambda_i \left(\mathbf{D} - \frac{d}{m} \mathbf{M} \right) \mathbf{v}_i = 0 \\
& \left(\mathbf{K} + \lambda_i \mathbf{D} - \left(\frac{k}{m} + \lambda_i \frac{d}{m} \right) \mathbf{M} \right) \mathbf{v}_i = \left(\mathbf{K} + \lambda_i \mathbf{D} + \lambda_i^2 \mathbf{M} \right) \mathbf{v}_i = 0
\end{aligned} \tag{C.12}$$

Equations (C.11) and (C.12) give equation (C.2).

Appendix D. Calculating eigenvalue, eigenvector and thier derivatives from diagonalised system matrices and DSPEs

Considering a general non symmetric damped system, the governing equation of motion of the second order system is

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f} \quad (\text{D.1})$$

where $\{\mathbf{K}, \mathbf{D}, \mathbf{M}\}$ are the system matrices which satisfy the following equations

$$(\lambda_i^2 \mathbf{M} + \lambda_i \mathbf{D} + \mathbf{K}) \phi_{Ri} = 0 \quad , \quad \phi_{Li}^T (\lambda_i^2 \mathbf{M} + \lambda_i \mathbf{D} + \mathbf{K}) = 0 \quad (\text{D.2})$$

The $\{\phi_{Ri}, \phi_{Li}\}$ are the i^{th} right and left eigenvectors, where $i = \{1, 2, \dots, n\}$

In state space, the right and left modal matrices $\{\underline{\Phi}_R, \underline{\Phi}_L\}$ are $(2n \times 2n)$. These diagonalise the Lancaster augmented matrices (LAMs) $\{\underline{\mathbf{K}}, \underline{\mathbf{D}}, \underline{\mathbf{M}}\}$. The eigenvalues $\underline{\Lambda}$ might be complex or real or mixed (real and complex). If all eigenvalues are complex $\underline{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_1^*, \lambda_2^*, \dots, \lambda_n^*)$ and assuming that all eigenvalues are “semi simple”

$$\begin{bmatrix} \Phi_L & \Phi_L^* \\ \Lambda \Phi_L & \Lambda^* \Phi_L^* \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \Phi_R & \Phi_R^* \\ \Lambda \Phi_R & \Lambda^* \Phi_R^* \end{bmatrix} = \begin{bmatrix} \Lambda \Psi & \mathbf{0} \\ \mathbf{0} & \Lambda^* \Psi^* \end{bmatrix} = \begin{bmatrix} \Omega & \mathbf{0} \\ \mathbf{0} & \Omega^* \end{bmatrix} \quad (\text{D.3})$$

$$\begin{bmatrix} \Phi_L & \Phi_L^* \\ \Lambda \Phi_L & \Lambda^* \Phi_L^* \end{bmatrix}^T \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \Phi_R & \Phi_R^* \\ \Lambda \Phi_R & \Lambda^* \Phi_R^* \end{bmatrix} = \begin{bmatrix} \Psi & \mathbf{0} \\ \mathbf{0} & \Psi^* \end{bmatrix} \quad (\text{D.4})$$

$$\begin{bmatrix} \Phi_L & \Phi_L^* \\ \Lambda \Phi_L & \Lambda^* \Phi_L^* \end{bmatrix}^T \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Phi_R & \Phi_R^* \\ \Lambda \Phi_R & \Lambda^* \Phi_R^* \end{bmatrix} = \begin{bmatrix} \Psi/\Lambda & \mathbf{0} \\ \mathbf{0} & \Psi^*/\Lambda^* \end{bmatrix} = \begin{bmatrix} \Gamma & \mathbf{0} \\ \mathbf{0} & \Gamma^* \end{bmatrix} \quad (\text{D.5})$$

Equations (5.73), (5.74) and (5.75) can be rewritten as

$$\underline{\Phi}_L^T \underline{\mathbf{M}} \underline{\Phi}_R = \underline{\Omega} \quad (\text{D.6})$$

$$\underline{\Phi}_L^T \underline{\mathbf{D}} \underline{\Phi}_R = \underline{\Psi} \quad (\text{D.7})$$

$$\underline{\Phi}_L^T \underline{\mathbf{K}} \underline{\Phi}_R = \underline{\Gamma} \quad (\text{D.8})$$

where $\{\underline{\Omega}, \underline{\Psi}, \underline{\Gamma}\}$ are diagonal matrices $\underline{\Omega} = \text{diag}(\omega_1, \omega_2, \dots, \omega_n)$, $\underline{\Psi} = \text{diag}(\psi_1, \psi_2, \dots, \psi_n)$ and $\underline{\Gamma} = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_n)$. If the left and right matrices of long eigenvectors $\{\underline{\Phi}_L, \underline{\Phi}_R\}$ are normalised using equation (5.74) and setting $\underline{\Psi} = \underline{\mathbf{I}}$, the values of $\{\underline{\Omega}, \underline{\Psi}, \underline{\Gamma}\}$ are still diagonal and can be expressed as $\{\underline{\Lambda}, \underline{\mathbf{I}}, \underline{\Lambda}^{-1}\}$ respectively.

We will select two columns of eigenvectors at a time: the i^{th} column and the complex conjugate of the i^{th} column. Equations (5.73), (5.74) and (5.75) can be written as

$$\begin{bmatrix} \phi_{Li} & \phi_{Li}^* \\ \lambda_i \phi_{Li} & \lambda_i^* \phi_{Li}^* \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \phi_{Ri} & \phi_{Ri}^* \\ \lambda_i \phi_{Ri} & \lambda_i^* \phi_{Ri}^* \end{bmatrix} = \begin{bmatrix} \lambda_i \psi_i & \mathbf{0} \\ \mathbf{0} & \lambda_i^* \psi_i^* \end{bmatrix} = \begin{bmatrix} \omega_i & \mathbf{0} \\ \mathbf{0} & \omega_i^* \end{bmatrix} \quad (\text{D.9})$$

$$\begin{bmatrix} \phi_{Li} & \phi_{Li}^* \\ \lambda_i \phi_{Li} & \lambda_i^* \phi_{Li}^* \end{bmatrix}^T \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \phi_{Ri} & \phi_{Ri}^* \\ \lambda_i \phi_{Ri} & \lambda_i^* \phi_{Ri}^* \end{bmatrix} = \begin{bmatrix} \psi_i & \mathbf{0} \\ \mathbf{0} & \psi_i^* \end{bmatrix} \quad (\text{D.10})$$

$$\begin{bmatrix} \phi_{Li} & \phi_{Li}^* \\ \lambda_i \phi_{Li} & \lambda_i^* \phi_{Li}^* \end{bmatrix}^T \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \phi_{Ri} & \phi_{Ri}^* \\ \lambda_i \phi_{Ri} & \lambda_i^* \phi_{Ri}^* \end{bmatrix} = \begin{bmatrix} \psi_i / \lambda_i & \mathbf{0} \\ \mathbf{0} & \psi_i^* / \lambda_i^* \end{bmatrix} = \begin{bmatrix} \gamma_i & \mathbf{0} \\ \mathbf{0} & \gamma_i^* \end{bmatrix} \quad (\text{D.11})$$

More concisely

$$\underline{\phi}_{Li}^T \underline{\mathbf{M}} \underline{\phi}_{Ri} = \omega_i \quad (\text{D.12})$$

$$\underline{\phi}_{Li}^T \underline{\mathbf{D}} \underline{\phi}_{Ri} = \psi_i \quad (\text{D.13})$$

$$\underline{\phi}_{Li}^T \underline{\mathbf{K}} \underline{\phi}_{Ri} = \gamma_i \quad (\text{D.14})$$

It is possible to find real-valued left and right diagonalising transformations $\{\underline{\mathbf{T}}_L, \underline{\mathbf{T}}_R\}$ which transform the system matrices into block diagonal matrices having an identical spectrum

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_D \\ \mathbf{K}_D & \mathbf{D}_D \end{bmatrix} \quad (\text{D.15})$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{K}_D & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_D \end{bmatrix} \quad (\text{D.16})$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} -\mathbf{D}_D & -\mathbf{M}_D \\ -\mathbf{M}_D & \mathbf{0} \end{bmatrix} \quad (\text{D.17})$$

Equations (5.86), (5.87) and (5.88) can be rewritten as

$$\underline{\mathbf{T}}_L^T \underline{\mathbf{M}} \underline{\mathbf{T}}_R = \underline{\mathbf{M}}_D \quad (\text{D.18})$$

$$\underline{\mathbf{T}}_L^T \underline{\mathbf{D}} \underline{\mathbf{T}}_R = \underline{\mathbf{D}}_D \quad (\text{D.19})$$

$$\underline{\mathbf{T}}_L^T \underline{\mathbf{K}} \underline{\mathbf{T}}_R = \underline{\mathbf{K}}_D \quad (\text{D.20})$$

where $\{\underline{\mathbf{K}}_D, \underline{\mathbf{D}}_D, \underline{\mathbf{M}}_D\}$ are block diagonal matrices and $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$ are diagonal matrices in the form

$$\begin{aligned} \mathbf{K}_D &= \text{diag}(k_1, k_2, \dots, k_n) \\ \mathbf{D}_D &= \text{diag}(d_1, d_2, \dots, d_n) \\ \mathbf{M}_D &= \text{diag}(m_1, m_2, \dots, m_n) \end{aligned} \quad (\text{D.21})$$

If the left and right diagonalising transformations $\{\underline{\mathbf{T}}_L, \underline{\mathbf{T}}_R\}$ are in a particular way normalised, then the equation (2.77) can be written as

$$\begin{aligned} \mathbf{K}_D &= \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \\ \mathbf{D}_D &= \text{diag}(2\zeta_1\lambda_1, 2\zeta_2\lambda_2, \dots, 2\zeta_n\lambda_n) \\ \mathbf{M}_D &= \text{identity} \end{aligned} \quad (\text{D.22})$$

The left and right diagonalising transformations $\{\underline{\mathbf{T}}_{Li}, \underline{\mathbf{T}}_{Ri}\}$ satisfy the following equations

$$(m_i \underline{\mathbf{M}} + d_i \underline{\mathbf{D}} + k_i \underline{\mathbf{K}}) \underline{\mathbf{T}}_{Ri} = 0, \quad \underline{\mathbf{T}}_{Li}^T (m_i \underline{\mathbf{M}} + d_i \underline{\mathbf{D}} + k_i \underline{\mathbf{K}}) = 0 \quad (\text{D.23})$$

Select columns i and $(n+i)$ from the diagonalising transformations at a time.

Equations (5.86), (5.87) and (5.88) can be written as

$$\begin{bmatrix} w_{Li} & x_{Li} \\ y_{Li} & z_{Li} \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} w_{Ri} & x_{Ri} \\ y_{Ri} & z_{Ri} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & k_i \\ k_i & d_i \end{bmatrix} \quad (\text{D.24})$$

$$\begin{bmatrix} w_{Li} & x_{Li} \\ y_{Li} & z_{Li} \end{bmatrix}^T \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{bmatrix} w_{Ri} & x_{Ri} \\ y_{Ri} & z_{Ri} \end{bmatrix} = \begin{bmatrix} k_i & 0 \\ 0 & -m_i \end{bmatrix} \quad (\text{D.25})$$

$$\begin{bmatrix} w_{Li} & x_{Li} \\ y_{Li} & z_{Li} \end{bmatrix}^T \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} w_{Ri} & x_{Ri} \\ y_{Ri} & z_{Ri} \end{bmatrix} = \begin{bmatrix} -d_i & -m_i \\ -m_i & 0 \end{bmatrix} \quad (\text{D.26})$$

We can simplify equations (5.106), (5.107) and (5.108) as

$$\underline{\mathbf{T}}_{Li}^T \underline{\mathbf{M}} \underline{\mathbf{T}}_{Ri} = \underline{m}_i \quad (\text{D.27})$$

$$\underline{\mathbf{T}}_{Li}^T \underline{\mathbf{D}} \underline{\mathbf{T}}_{Ri} = \underline{d}_i \quad (\text{D.28})$$

$$\underline{\mathbf{T}}_{Li}^T \underline{\mathbf{K}} \underline{\mathbf{T}}_{Ri} = \underline{k}_i \quad (\text{D.29})$$

D.1 *Calculating pairs of eigenvalues and their derivatives from the diagonalised system matrices and its derivatives*

The i^{th} eigenvalue $\{\lambda_{i1}, \lambda_{i2}\}$ can be calculated from the i^{th} diagonalised system $\{k_i, d_i, m_i\}$ using

$$(\lambda_i^2 m_i + \lambda_i d_i + k_i) = 0 \quad (\text{D.30})$$

It is possible to find the roots of λ_i from equation (D.30) and through

$$\lambda_{i1,2} = \frac{-d_i \pm \sqrt{d_i^2 - 4k_i m_i}}{2m_i} \quad (\text{D.31})$$

Differentiating equation (D.30) with respect to θ gives an expression for the eigenvalue derivatives

$$\frac{\partial \lambda_{i1,2}}{\partial \theta} = \dot{\lambda}_{i1,2} = \frac{-(\dot{k}_i + \dot{d}_i \lambda_{i1,2} + \dot{m}_i \lambda_{i1,2}^2)}{(d_i + 2m_i \lambda_{i1,2})} \quad (\text{D.32})$$

D.2 Calculating two columns of eigenvectors and their derivatives from DSPEs and its derivatives

Comparing equations (5.79), (5.80) and (5.81) with equations (5.106), (5.107) and (5.108) yields

$$\begin{bmatrix} \lambda_i \psi_i & \mathbf{0} \\ \mathbf{0} & \lambda_i^* \psi_i^* \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & k_i \\ k_i & d_i \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad (\text{D.33})$$

$$\begin{bmatrix} \psi_i & \mathbf{0} \\ \mathbf{0} & \psi_i^* \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^T \begin{bmatrix} k_i & 0 \\ 0 & -m_i \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad (\text{D.34})$$

$$\begin{bmatrix} \psi_i / \lambda_i & \mathbf{0} \\ \mathbf{0} & \psi_i^* / \lambda_i^* \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^T \begin{bmatrix} -d_i & -m_i \\ -m_i & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad (\text{D.35})$$

More concisely

$$\underline{\omega}_i = \underline{\mathbf{a}}_i^T \underline{\mathbf{T}}_{Li}^T \underline{m}_i \underline{\mathbf{T}}_{Ri} \underline{\mathbf{a}}_i \quad (\text{D.36})$$

$$\underline{\psi}_i = \underline{\mathbf{a}}_i^T \underline{\mathbf{T}}_{Li}^T \underline{d}_i \underline{\mathbf{T}}_{Ri} \underline{\mathbf{a}}_i \quad (\text{D.37})$$

$$\underline{\gamma}_i = \underline{\mathbf{a}}_i^T \underline{\mathbf{T}}_{Li}^T \underline{k}_i \underline{\mathbf{T}}_{Ri} \underline{\mathbf{a}}_i \quad (\text{D.38})$$

Equation (5.157), (5.158) and (5.159) show that the (2×2) matrix $\underline{\mathbf{a}}_i$ is the eigenvector matrix of $\{k_i, d_i, m_i\}$. Scaling the matrix $\underline{\mathbf{a}}_i$ as

$$\underline{\mathbf{a}}_i = \underline{\mathbf{a}}_i / \text{diag}([\underline{\mathbf{a}}_i(1,1) \quad \underline{\mathbf{a}}_i(1,2)]) \quad (\text{D.39})$$

The matrix $\underline{\mathbf{a}}_i$ can be written as

$$\underline{\mathbf{a}}_i = \begin{bmatrix} 1 & 1 \\ \lambda_i & \lambda_i^* \end{bmatrix} \quad (\text{D.40})$$

The left and right columns i and $(n+i)$ of the eigenvector of the original system can be calculated from the left and right columns i and $(n+i)$ of the diagonalising transformations as

$$\begin{bmatrix} \phi_{Ri} & \phi_{Ri}^* \\ \lambda_i \phi_{Ri} & \lambda_i^* \phi_{Ri}^* \end{bmatrix} = \begin{bmatrix} w_{Ri} & x_{Ri} \\ y_{Ri} & z_{Ri} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \lambda_i & \lambda_i^* \end{bmatrix} \quad (\text{D.41})$$

$$\begin{bmatrix} \phi_{Li} & \phi_{Li}^* \\ \lambda_i \phi_{Li} & \lambda_i^* \phi_{Li}^* \end{bmatrix} = \begin{bmatrix} w_{Li} & x_{Li} \\ y_{Li} & z_{Li} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \lambda_i & \lambda_i^* \end{bmatrix} \quad (\text{D.42})$$

More concisely

$$\underline{\phi_{Ri}} = \underline{\mathbf{T}_{Ri}} \underline{\mathbf{a}_i} \quad (\text{D.43})$$

$$\underline{\phi_{Li}} = \underline{\mathbf{T}_{Li}} \underline{\mathbf{a}_i} \quad (\text{D.44})$$

Differentiating equations (5.164) and (5.165) with respect to $\boldsymbol{\theta}$ gives the expressions for the eigenvector derivatives as

$$\frac{\partial \underline{\phi_{Ri}}}{\partial \boldsymbol{\theta}} = \underline{\dot{\phi}_{Ri}} = \underline{\dot{\mathbf{T}}_{Ri}} \underline{\mathbf{a}_i} + \underline{\mathbf{T}_{Ri}} \underline{\dot{\mathbf{a}}_i} \quad (\text{D.45})$$

$$\frac{\partial \underline{\phi_{Li}}}{\partial \boldsymbol{\theta}} = \underline{\dot{\phi}_{Li}} = \underline{\dot{\mathbf{T}}_{Li}} \underline{\mathbf{a}_i} + \underline{\mathbf{T}_{Li}} \underline{\dot{\mathbf{a}}_i} \quad (\text{D.46})$$

Where the derivative of $\underline{\mathbf{a}_i}$ can be written as

$$\underline{\dot{\mathbf{a}}_i} = \begin{bmatrix} 0 & 0 \\ \dot{\lambda}_i & \dot{\lambda}_i^* \end{bmatrix} \quad (\text{D.47})$$

Appendix E. Extracting the measured DSPEs using Modal filters based on Möbius transformation

The right and left diagonalising SPEs $\{\underline{\mathbf{T}}_{Ro}, \underline{\mathbf{T}}_{Lo}\}$ for the original system $\{\mathbf{M}_o, \mathbf{D}_o, \mathbf{K}_o\}$ can be extracted from the diagonalising SPEs $\{\underline{\mathbf{T}}_{RN}, \underline{\mathbf{T}}_{LN}\}$ for the new system $\{\mathbf{M}_N, \mathbf{D}_N, \mathbf{K}_N\}$ using the spectral transformations which are discussed in chapter 6.

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_D \\ \mathbf{K}_D & \mathbf{D}_D \end{bmatrix} \quad (\text{E.1})$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{K}_D & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_D \end{bmatrix} \quad (\text{E.2})$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} -\mathbf{D}_D & -\mathbf{M}_D \\ -\mathbf{M}_D & \mathbf{0} \end{bmatrix} \quad (\text{E.3})$$

The direct relation of Möbius transformation is

$$\lambda = \frac{p\gamma + q}{r\gamma + s} \quad (\text{E.4})$$

The inverse of Möbius transformations in equation (E.4) is

$$\gamma = \frac{s\lambda - q}{-r\lambda + p} \quad (\text{E.5})$$

The diagonalised system matrices are calculated

$$\begin{bmatrix} \mathbf{K}_{Do} \\ \mathbf{D}_{Do} \\ \mathbf{M}_{Do} \end{bmatrix} = (\mathbf{Q}_{SSTO} \otimes \mathbf{I}_n) \begin{bmatrix} \mathbf{K}_{DN} \\ \mathbf{D}_{DN} \\ \mathbf{M}_{DN} \end{bmatrix} \quad (\text{E.6})$$

Houlston [145] presents the right and left polynomial of SPEs based modal filters.

The SPEs are used to transform the original *LAMs* matrices to the block diagonal *LAMs* matrices. Then, SPEs modal filters can be written as

$$[\mathbf{I} \quad \gamma \mathbf{I}](\underline{\mathbf{D}}_N - \gamma \underline{\mathbf{K}}_N) \underline{\mathbf{T}}_{RN} \begin{bmatrix} \mathbf{I} \\ \gamma \mathbf{I} \end{bmatrix} = [\mathbf{I} \quad \gamma \mathbf{I}] \underline{\mathbf{T}}_{LN}^{-T} (\underline{\mathbf{D}}_{DN} - \gamma \underline{\mathbf{K}}_{DN}) \begin{bmatrix} \mathbf{I} \\ \gamma \mathbf{I} \end{bmatrix} \quad (\text{E.7})$$

$$[\mathbf{I} \quad \gamma \mathbf{I}](\underline{\mathbf{M}}_N - \gamma \underline{\mathbf{D}}_N) \underline{\mathbf{T}}_{RN} \begin{bmatrix} \mathbf{I} \\ \gamma \mathbf{I} \end{bmatrix} = [\mathbf{I} \quad \gamma \mathbf{I}] \underline{\mathbf{T}}_{LN}^{-T} (\underline{\mathbf{M}}_{DN} - \gamma \underline{\mathbf{D}}_{DN}) \begin{bmatrix} \mathbf{I} \\ \gamma \mathbf{I} \end{bmatrix} \quad (\text{E.8})$$

where

$$\underline{\mathbf{T}}_{LN}^{-1} = \begin{bmatrix} \mathbf{W}_{LN} & \mathbf{X}_{LN} \\ \mathbf{Y}_{LN} & \mathbf{Z}_{LN} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{W}_{SN} & \mathbf{X}_{SN} \\ \mathbf{Y}_{SN} & \mathbf{Z}_{SN} \end{bmatrix} \quad (\text{E.9})$$

For a second order system having system matrices $\{\mathbf{M}_N, \mathbf{D}_N, \mathbf{K}_N\}$, it is possible to find first order filters $(\mathbf{W}_{RN} + \gamma_i \mathbf{X}_{RN})$ and $(\mathbf{W}_{SN} + \gamma_i \mathbf{X}_{SN})$ or $(\mathbf{Y}_{RN} + \gamma_i \mathbf{Z}_{RN})$ and $(\mathbf{Y}_{SN} + \gamma_i \mathbf{Z}_{SN})$ which diagonalise the system matrices into diagonal $(N \times N)$ matrices $\{\mathbf{M}_{DN}, \mathbf{D}_{DN}, \mathbf{K}_{DN}\}$. The diagonalising filters satisfy these equations for all values of γ

$$(\mathbf{K}_N + \gamma_i \mathbf{D}_N + \gamma_i^2 \mathbf{M}_N)(\mathbf{W}_{RN} + \gamma_i \mathbf{X}_{RN}) = (\mathbf{W}_{SN} + \gamma_i \mathbf{X}_{SN})^T (\mathbf{K}_{DN} + \gamma_i \mathbf{D}_{DN} + \gamma_i^2 \mathbf{M}_{DN}) \quad (\text{E.10})$$

$$(\mathbf{K}_N + \gamma_i \mathbf{D}_N + \gamma_i^2 \mathbf{M}_N)(\mathbf{Y}_{RN} + \gamma_i \mathbf{Z}_{RN}) = (\mathbf{Y}_{SN} + \gamma_i \mathbf{Z}_{SN})^T (\mathbf{K}_{DN} + \gamma_i \mathbf{D}_{DN} + \gamma_i^2 \mathbf{M}_{DN}) \quad (\text{E.11})$$

To choose which of the above equations can be used it depends on the original system matrices. In some cases, the original mass matrix is singular then equation (E.11) can be chosen to calculate the original system diagonalising transformations.

Substituting equation (6.8) into equation (E.11) and dividing by $(qr - ps)$ yields

$$(\mathbf{K}_o + \lambda_i \mathbf{D}_o + \lambda_i^2 \mathbf{M}_o)(\mathbf{Y}_{Ro} + \lambda_i \mathbf{Z}_{Ro}) = (\mathbf{Y}_{So} + \lambda_i \mathbf{Z}_{So})^T (\mathbf{K}_{Do} + \lambda_i \mathbf{D}_{Do} + \lambda_i^2 \mathbf{M}_{Do}) \quad (\text{E.12})$$

in which

$$\begin{aligned} \mathbf{Y}_{Ro} &= (p\mathbf{Y}_{RN} - q\mathbf{Z}_{RN}) \quad , \quad \mathbf{Y}_{So} = (p\mathbf{Y}_{SN} - q\mathbf{Z}_{SN}) \\ \mathbf{Z}_{Ro} &= (-r\mathbf{Y}_{RN} + s\mathbf{Z}_{RN}) \quad , \quad \mathbf{Z}_{So} = (-r\mathbf{Y}_{SN} + s\mathbf{Z}_{SN}) \end{aligned} \quad (\text{E.13})$$

The diagonalised system matrices are calculated from equation (E.6). The right diagonalising transformation $\underline{\mathbf{T}}_{Ro}$ for the original system can be shown in the form as

$$\underline{\mathbf{T}}_{Ro} = \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} = \begin{bmatrix} \left(\mathbf{F}_{Ro} - \frac{1}{2}\mathbf{G}_{Ro}\mathbf{D}_{Do}\right) & (-\mathbf{G}_{Ro}\mathbf{M}_{Do}) \\ (\mathbf{G}_{Ro}\mathbf{K}_{Do}) & \left(\mathbf{F}_{Ro} + \frac{1}{2}\mathbf{G}_{Ro}\mathbf{D}_{Do}\right) \end{bmatrix} \quad (\text{E.14})$$

using $\{\mathbf{Y}_{Ro}, \mathbf{Z}_{Ro}\}$ in equation (E.12) to extract the $\{\mathbf{G}_{Ro}, \mathbf{F}_{Ro}\}$ and then to calculate $\{\mathbf{W}_{Ro}, \mathbf{X}_{Ro}\}$

$$\mathbf{G}_{Ro} = \mathbf{Y}_{Ro}\mathbf{K}_{Do}^{-1} \quad (\text{E.15})$$

$$\mathbf{F}_{Ro} = \mathbf{Z}_{Ro} - \frac{1}{2}\mathbf{G}_{Ro}\mathbf{D}_{Do} \quad (\text{E.16})$$

The $\{\mathbf{W}_{Ro}, \mathbf{X}_{Ro}\}$ can be calculated by substituting $\{\mathbf{G}_{Ro}, \mathbf{F}_{Ro}\}$ into the following equations

$$\mathbf{W}_{Ro} = \mathbf{F}_{Ro} - \frac{1}{2}\mathbf{G}_{Ro}\mathbf{D}_{Do} \quad (\text{E.17})$$

$$\mathbf{X}_{Ro} = -\mathbf{G}_{Ro}\mathbf{M}_{Do} \quad (\text{E.18})$$

The left diagonalising transformation $\underline{\mathbf{T}}_{Lo}$ can be calculated using (E.1).

$$\underline{\mathbf{T}}_{Lo} = \begin{bmatrix} \mathbf{W}_{Lo} & \mathbf{X}_{Lo} \\ \mathbf{Y}_{Lo} & \mathbf{Z}_{Lo} \end{bmatrix} = \left(\begin{bmatrix} \mathbf{0} & \mathbf{K}_{Do} \\ \mathbf{K}_{Do} & \mathbf{D}_{Do} \end{bmatrix} \left(\begin{bmatrix} \mathbf{0} & \mathbf{K}_o \\ \mathbf{K}_o & \mathbf{D}_o \end{bmatrix} \begin{bmatrix} \mathbf{W}_{Ro} & \mathbf{X}_{Ro} \\ \mathbf{Y}_{Ro} & \mathbf{Z}_{Ro} \end{bmatrix} \right)^{-1} \right)^T \quad (\text{E.19})$$

The above approach for a singular mass matrix is summarised into these few steps :-

1. Transform the eigenvalue problem for the original system into a new system using the Mobius transformations equation (6.7) and divide the equation by $(qr - ps)$.
2. Find the right and left diagonalising transformations $\{\underline{\mathbf{T}}_{RN}, \underline{\mathbf{T}}_{LN}\}$ for the new system.
3. Calculate the diagonalised system matrices $\{\mathbf{M}_{Do}, \mathbf{D}_{Do}, \mathbf{K}_{Do}\}$ using equation (E.6).

4. Extract the original systems right and left diagonalising SPEs $\{\underline{\mathbf{T}}_{Ro}, \underline{\mathbf{T}}_{Lo}\}$ from the new system using SPEs based modal filters equation (E.8)
 - Substitute equation (6.8) into equation (E.11) which transforms the new system back into the original system, equation (E.12).
 - Calculate the right diagonalising SPEs $\underline{\mathbf{T}}_{Ro}$.
 - First, extract $\{\mathbf{G}_{Ro}, \mathbf{F}_{Ro}\}$ from $\{\mathbf{Y}_{Ro}, \mathbf{Z}_{Ro}\}$ using equations (E.13), (E.15) and (E.16).
 - Second, substitute the values of $\{\mathbf{G}_{Ro}, \mathbf{F}_{Ro}\}$ into equations (E.17) and (E.18) to calculate $\{\mathbf{W}_{Ro}, \mathbf{X}_{Ro}\}$.
 - Calculate the left diagonalising transformation $\underline{\mathbf{T}}_{Lo}$ as in equation (E.19) by using equation (E.1).

Appendix F. Calculating measured DSPTs using homogenous coordinates based on Möbius transformation

Diagonalising SPTs are represented by real-valued ($n \times n$) matrices $\{\mathbf{W}_L, \mathbf{X}_L, \mathbf{Y}_L, \mathbf{Z}_L\}$, $\{\mathbf{W}_R, \mathbf{X}_R, \mathbf{Y}_R, \mathbf{Z}_R\}$. These transformations $\{\underline{\mathbf{T}}_L, \underline{\mathbf{T}}_R\}$ can usually decouple the equations of motion such that the new system matrices $\{\mathbf{K}_D, \mathbf{D}_D, \mathbf{M}_D\}$ themselves are diagonal.

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_D \\ \mathbf{K}_D & \mathbf{D}_D \end{bmatrix} \quad (\text{F.1})$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} \mathbf{K}_D & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_D \end{bmatrix} \quad (\text{F.2})$$

$$\begin{bmatrix} \mathbf{W}_L & \mathbf{X}_L \\ \mathbf{Y}_L & \mathbf{Z}_L \end{bmatrix}^T \begin{bmatrix} -\mathbf{D} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W}_R & \mathbf{X}_R \\ \mathbf{Y}_R & \mathbf{Z}_R \end{bmatrix} = \begin{bmatrix} -\mathbf{D}_D & -\mathbf{M}_D \\ -\mathbf{M}_D & \mathbf{0} \end{bmatrix} \quad (\text{F.3})$$

The homogenous coordinate expression defines a pair of eigenvalues and the associated pair of eigenvectors. For the original system $\{\mathbf{M}_o, \mathbf{D}_o, \mathbf{K}_o\}$, the homogenous coordinate can be written as

$$\left[m_{oi} \begin{bmatrix} 0 & \mathbf{K}_o \\ \mathbf{K}_o & \mathbf{D}_o \end{bmatrix} + d_{oi} \begin{bmatrix} \mathbf{K}_o & 0 \\ 0 & -\mathbf{M}_o \end{bmatrix} + k_{oi} \begin{bmatrix} -\mathbf{D}_o & -\mathbf{M}_o \\ -\mathbf{M}_o & 0 \end{bmatrix} \right] \begin{bmatrix} \mathbf{w}_{Roi} & \mathbf{x}_{Roi} \\ \mathbf{y}_{Roi} & \mathbf{z}_{Roi} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (\text{F.4})$$

where $\{m_{oi}, d_{oi}, k_{oi}\}$ are the i^{th} entries of the diagonalised system $\{\mathbf{M}_{Do}, \mathbf{D}_{Do}, \mathbf{K}_{Do}\}$ and $\{\mathbf{w}_{Roi}, \mathbf{x}_{Roi}, \mathbf{y}_{Roi}, \mathbf{z}_{Roi}\}$ are the i^{th} column vectors of the respective block of right diagonalising SPE of the original system. For some vectors, $\{\mathbf{f}_{Roi}, \mathbf{g}_{Roi}\}$

$$\mathbf{w}_{Roi} = \mathbf{f}_{Roi} - \frac{1}{2} \mathbf{g}_{Roi} d_{oi} \quad (\text{F.5})$$

$$\mathbf{x}_{Roi} = -\mathbf{g}_{Roi} m_{oi} \quad (\text{F.6})$$

$$\mathbf{y}_{Roi} = \mathbf{g}_{Roi} k_{oi} \quad (\text{F.7})$$

$$\mathbf{z}_{Roi} = \mathbf{f}_{Roi} + \frac{1}{2} \mathbf{g}_{Roi} d_{oi} \quad (\text{F.8})$$

Note that

$$\lambda^2 m_{oi} + \lambda d_{oi} + k_{oi} = 0 \quad (\text{F.9})$$

Substituting equations (F.5)-(F.8) into equation (F.4) yields four homogenous vector equations

$$\left(d_{oi} \mathbf{K}_o \mathbf{f}_{Roi} + \left(m_{oi} k_{oi} - \frac{1}{2} d_{oi}^2 \right) \mathbf{K}_o \mathbf{g}_{Roi} - k_{oi} \mathbf{D}_o \mathbf{f}_{Roi} + \frac{1}{2} k_{oi} d_{oi} \mathbf{D}_o \mathbf{g}_{Roi} - k_{oi}^2 \mathbf{M}_o \mathbf{g}_{Roi} \right) = 0 \quad (\text{F.10})$$

$$\left(m_{oi} \mathbf{K}_o \mathbf{f}_{Roi} - \frac{1}{2} d_{oi} m_{oi} \mathbf{K}_o \mathbf{g}_{Roi} - k_{oi} m_{oi} \mathbf{D}_o \mathbf{g}_{Roi} - k_{oi} \mathbf{M}_o \mathbf{f}_{Roi} - \frac{1}{2} k_{oi} d_{oi} \mathbf{M}_o \mathbf{g}_{Roi} \right) = 0 \quad (\text{F.11})$$

$$\left(m_{oi} \mathbf{K}_o \mathbf{f}_{Roi} - \frac{1}{2} d_{oi} m_{oi} \mathbf{K}_o \mathbf{g}_{Roi} - k_{oi} m_{oi} \mathbf{D}_o \mathbf{g}_{Roi} - k_{oi} \mathbf{M}_o \mathbf{f}_{Roi} - \frac{1}{2} k_{oi} d_{oi} \mathbf{M}_o \mathbf{g}_{Roi} \right) = 0 \quad (\text{F.12})$$

$$\left(-m_{oi}^2 \mathbf{K}_o \mathbf{g}_{Roi} + m_{oi} \mathbf{D}_o \mathbf{f}_{Roi} + \frac{1}{2} m_{oi} d_{oi} \mathbf{D}_o \mathbf{g}_{Roi} - d_{oi} \mathbf{M}_o \mathbf{f}_{Roi} + \left(k_{oi} m_{oi} - \frac{1}{2} d_{oi}^2 \right) \mathbf{M}_o \mathbf{g}_{Roi} \right) = 0 \quad (\text{F.13})$$

Equations (F.11) and (F.12) are identical. Equations (F.10), (F.11) and (F.13) produce $(3n \times 1)$ homogenous equations

$$\left(\begin{bmatrix} d_{oi} & \left(m_{oi} k_{oi} - \frac{1}{2} d_{oi}^2 \right) & -k_{oi} & \frac{1}{2} k_{oi} d_{oi} & 0 & -k_{oi}^2 \\ m_{oi} & -\frac{1}{2} d_{oi} m_{oi} & 0 & k_{oi} m_{oi} & -k_{oi} & -\frac{1}{2} k_{oi} d_{oi} \\ 0 & -m_{oi}^2 & m_{oi} & \frac{1}{2} m_{oi} d_{oi} & -d_{oi} & \left(k_{oi} m_{oi} - \frac{1}{2} d_{oi}^2 \right) \end{bmatrix} \otimes \mathbf{I}_n \right) \mathbf{B}_o \begin{bmatrix} \mathbf{f}_{Roi} \\ \mathbf{g}_{Roi} \end{bmatrix} = 0 \quad (\text{F.14})$$

where \otimes represents the Kronecker product and \mathbf{I}_n is the $(n \times n)$ identity matrix and

$$\mathbf{B}_o = \begin{bmatrix} \mathbf{K}_o & 0 \\ 0 & \mathbf{K}_o \\ \mathbf{D}_o & 0 \\ 0 & \mathbf{D}_o \\ \mathbf{M}_o & 0 \\ 0 & \mathbf{M}_o \end{bmatrix} \quad (\text{F.15})$$

Applying the Mobius transformation

$$\lambda = \frac{p\gamma + q}{r\gamma + s} \quad (\text{F.16})$$

to obtain a new second order system $\{\mathbf{K}_N, \mathbf{D}_N, \mathbf{M}_N\}$ having the same short eigenvectors as the original system and eigenvalues related by (F.16). Note that

$$\begin{bmatrix} \mathbf{K}_o \\ \mathbf{D}_o \\ \mathbf{M}_o \end{bmatrix} = (\mathbf{Q}_{SSTO} \otimes \mathbf{I}_n) \begin{bmatrix} \mathbf{K}_N \\ \mathbf{D}_N \\ \mathbf{M}_N \end{bmatrix} \quad (\text{F.17})$$

$$\begin{bmatrix} \mathbf{K}_N \\ \mathbf{D}_N \\ \mathbf{M}_N \end{bmatrix} = (\mathbf{Q}_{SSTN} \otimes \mathbf{I}_n) \begin{bmatrix} \mathbf{K}_o \\ \mathbf{D}_o \\ \mathbf{M}_o \end{bmatrix} \quad (\text{F.18})$$

The diagonalised system matrices for the system $\{\mathbf{M}_o, \mathbf{D}_o, \mathbf{K}_o\}$ are

$$\begin{bmatrix} \mathbf{K}_{Do} \\ \mathbf{D}_{Do} \\ \mathbf{M}_{Do} \end{bmatrix} = (\mathbf{Q}_{SSTO} \otimes \mathbf{I}_n) \begin{bmatrix} \mathbf{K}_{DN} \\ \mathbf{D}_{DN} \\ \mathbf{M}_{DN} \end{bmatrix} \quad (\text{F.19})$$

where $\mathbf{Q}_{SSTN} = \mathbf{Q}_{SSTO}^{-1}$

$$\mathbf{Q}_{SSTN} = \frac{1}{(qr - ps)} \begin{bmatrix} s^2 & qs & q^2 \\ 2rs & (ps + qr) & 2pq \\ r^2 & rp & p^2 \end{bmatrix} \quad (\text{F.20})$$

Substituting equation (F.17) into equation (F.14), yields

$$(\mathbf{A}_o \otimes \mathbf{I}_n)(\mathbf{Q}_{SSTO} \otimes \mathbf{I}_{2n})\mathbf{B}_N \begin{bmatrix} \mathbf{f}_{Roi} \\ \mathbf{g}_{Roi} \end{bmatrix} = 0 \quad (\text{F.21})$$

where

$$\mathbf{A}_o = \begin{bmatrix} d_{oi} & \left(m_{oi}k_{oi} - \frac{1}{2}d_{oi}^2\right) & -k_{oi} & \frac{1}{2}k_{oi}d_{oi} & 0 & -k_{oi}^2 \\ m_{oi} & -\frac{1}{2}d_{oi}m_{oi} & 0 & k_{oi}m_{oi} & -k_{oi} & -\frac{1}{2}k_{oi}d_{oi} \\ 0 & -m_{oi}^2 & m_{oi} & \frac{1}{2}m_{oi}d_{oi} & -d_{oi} & \left(k_{oi}m_{oi} - \frac{1}{2}d_{oi}^2\right) \end{bmatrix} \quad (\text{F.22})$$

$$\mathbf{B}_N = \begin{bmatrix} \mathbf{K}_N & 0 \\ 0 & \mathbf{K}_N \\ \mathbf{D}_N & 0 \\ 0 & \mathbf{D}_N \\ \mathbf{M}_N & 0 \\ 0 & \mathbf{M}_N \end{bmatrix} \quad (\text{F.23})$$

The homogenous coordinate definition for the eigenvalues for the new system

$\{\mathbf{M}_N, \mathbf{D}_N, \mathbf{K}_N\}$ can be written as

$$\begin{bmatrix} m_{Ni} & \begin{bmatrix} 0 & \mathbf{K}_N \\ \mathbf{K}_N & \mathbf{D}_N \end{bmatrix} \end{bmatrix} + d_{Ni} \begin{bmatrix} \mathbf{K}_N & 0 \\ 0 & -\mathbf{M}_N \end{bmatrix} + k_{Ni} \begin{bmatrix} -\mathbf{D}_N & -\mathbf{M}_N \\ -\mathbf{M}_N & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w}_{RNi} & \mathbf{x}_{RNi} \\ \mathbf{y}_{RNi} & \mathbf{z}_{RNi} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (\text{F.24})$$

$$(\gamma^2 m_{Ni} + \gamma d_{Ni} + k_{Ni}) = 0 \quad (\text{F.25})$$

and there are vectors $\{\mathbf{f}_{RNi}, \mathbf{g}_{RNi}\}$ such that

$$\mathbf{w}_{RNi} = \mathbf{f}_{RNi} - \frac{1}{2} \mathbf{g}_{RNi} d_{Ni} \quad (\text{F.26})$$

$$\mathbf{x}_{RNi} = -\mathbf{g}_{RNi} m_{Ni} \quad (\text{F.27})$$

$$\mathbf{y}_{RNi} = \mathbf{g}_{RNi} k_{Ni} \quad (\text{F.28})$$

$$\mathbf{z}_{RNi} = \mathbf{f}_{RNi} + \frac{1}{2} \mathbf{g}_{RNi} d_{Ni} \quad (\text{F.29})$$

Equation (F.24) produces a set of equations in the same form as equation (F.14)

$$\left(\begin{bmatrix} d_{Ni} & \left(m_{Ni} k_{Ni} - \frac{1}{2} d_{Ni}^2\right) & -k_{Ni} & \frac{1}{2} k_{Ni} d_{Ni} & 0 & -k_{Ni}^2 \\ m_{Ni} & -\frac{1}{2} d_{Ni} m_{Ni} & 0 & k_{Ni} m_{Ni} & -k_{Ni} & -\frac{1}{2} k_{Ni} d_{Ni} \\ 0 & -m_{Ni}^2 & m_{Ni} & \frac{1}{2} m_{Ni} d_{Ni} & -d_{Ni} & \left(k_{Ni} m_{Ni} - \frac{1}{2} d_{Ni}^2\right) \end{bmatrix} \otimes \mathbf{I}_n \right) \mathbf{B}_N \begin{bmatrix} \mathbf{f}_{RNi} \\ \mathbf{g}_{RNi} \end{bmatrix} = 0 \quad (\text{F.30})$$

Equation (F.30) can be written as

$$(\mathbf{A}_N \otimes \mathbf{I}_n) \mathbf{B}_N \begin{bmatrix} \mathbf{f}_{RNi} \\ \mathbf{g}_{RNi} \end{bmatrix} = 0 \quad (\text{F.31})$$

The subspace spanned by $\{\mathbf{f}_{Roi}, \mathbf{g}_{Roi}\}$ in equation (F.21) is the same as that spanned by

$\{\mathbf{f}_{RNi}, \mathbf{g}_{RNi}\}$ in equation (F.31). Then, vectors $\{\mathbf{f}_{Roi}, \mathbf{g}_{Roi}\}$ can be written as a linear

combination with some unknown scalars $\{\alpha_{ff}, \alpha_{fg}, \alpha_{gf}, \alpha_{gg}\}$ such that

$$\begin{bmatrix} \mathbf{f}_{Roi} \\ \mathbf{g}_{Roi} \end{bmatrix} = \left(\begin{bmatrix} \alpha_{ff} & \alpha_{fg} \\ \alpha_{gf} & \alpha_{gg} \end{bmatrix} \otimes \mathbf{I}_n \right) \begin{bmatrix} \mathbf{f}_{RNi} \\ \mathbf{g}_{RNi} \end{bmatrix} \quad (\text{F.32})$$

Substituting equation (F.32) into equation (F.21) gives

$$(\mathbf{A}_o \otimes \mathbf{I}_n)(\mathbf{Q}_o \otimes \mathbf{I}_{2n})\mathbf{B}_N \begin{bmatrix} \mathbf{f}_{RNi} \\ \mathbf{g}_{RNi} \end{bmatrix} = 0 \quad (\text{F.33})$$

where $\boldsymbol{\alpha} = \begin{bmatrix} \alpha_{ff} & \alpha_{fg} \\ \alpha_{gf} & \alpha_{gg} \end{bmatrix} \otimes \mathbf{I}_n$ and has dimensions $(2n \times 2n)$. The equation (F.33) can be

rewritten as

$$(\mathbf{A}_o \otimes \mathbf{I}_n)(\mathbf{Q}_o \otimes \mathbf{I}_{2n})\boldsymbol{\delta}\mathbf{B}_N \begin{bmatrix} \mathbf{f}_{RNi} \\ \mathbf{g}_{RNi} \end{bmatrix} = 0 \quad (\text{F.34})$$

where $\boldsymbol{\delta}$ has dimensions $(6n \times 6n)$

$$\boldsymbol{\delta} = \begin{bmatrix} \boldsymbol{\alpha} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\alpha} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\alpha} \end{bmatrix} \quad (\text{F.35})$$

Define

$$\mathbf{X} = (\mathbf{A}_o \otimes \mathbf{I}_n)(\mathbf{Q}_{SSTO} \otimes \mathbf{I}_{2n})\boldsymbol{\delta} \quad (\text{F.36})$$

$$\mathbf{Y} = (\mathbf{A}_N \otimes \mathbf{I}_n) \quad (\text{F.37})$$

Comparing equation (F.34) with equation (F.31) it is clear that \mathbf{X} and \mathbf{Y} are equivalent. This is also to say there exists some invertible \mathbf{Z} for which

$$\mathbf{X} = \mathbf{Z}\mathbf{Y} \quad (\text{F.38})$$

Denote by \mathbf{Y}^Δ the right pseudo inverse of \mathbf{Y} , $\mathbf{Y}^\Delta = \text{pinv}(\mathbf{Y})$. Then evidently

$$\mathbf{Z} = \mathbf{X}\mathbf{Y}^\Delta \quad (\text{F.39})$$

in which \mathbf{Z} is a $(3n \times 3n)$ invariable matrix. \mathbf{X} and \mathbf{Z} have each four components $(3n \times 6n)$ such that

$$\mathbf{X} = \alpha_{ff}\mathbf{X}_{ff} + \alpha_{fg}\mathbf{X}_{fg} + \alpha_{gf}\mathbf{X}_{gf} + \alpha_{gg}\mathbf{X}_{gg} \quad (\text{F.40})$$

and similarly

$$\mathbf{Z} = \alpha_{ff}\mathbf{Z}_{ff} + \alpha_{fg}\mathbf{Z}_{fg} + \alpha_{gf}\mathbf{Z}_{gf} + \alpha_{gg}\mathbf{Z}_{gg} \quad (\text{F.41})$$

The partitions $\{\mathbf{X}_{ff}, \mathbf{X}_{fg}, \mathbf{X}_{gf}, \mathbf{X}_{gg}\}$ and $\{\mathbf{Z}_{ff}, \mathbf{Z}_{fg}, \mathbf{Z}_{gf}, \mathbf{Z}_{gg}\}$ can be calculated in turn by setting the four unknown scalars $\{\alpha_{ff}, \alpha_{fg}, \alpha_{gf}, \alpha_{gg}\}$ to unity one at a time in equations (F.36) and (F.39), respectively.

Set back \mathbf{Z} into equation (F.38) which is a function of $\{\alpha_{ff}, \alpha_{fg}, \alpha_{gf}, \alpha_{gg}\}$. Then, the four unknown scalars $\{\alpha_{ff}, \alpha_{fg}, \alpha_{gf}, \alpha_{gg}\}$ are calculated from equations (F.40) and (F.38) such that

$$\mathbf{W} \begin{bmatrix} \alpha_{ff} \\ \alpha_{fg} \\ \alpha_{gf} \\ \alpha_{gg} \end{bmatrix} = 0 \quad (\text{F.42})$$

where \mathbf{W} has a dimension $(18N^2 \times 4)$ and

$$\mathbf{W} = \begin{bmatrix} \text{vec}(\mathbf{X}_{ff} - \mathbf{Z}_{ff} \mathbf{Y}^\Delta) & \text{vec}(\mathbf{X}_{fg} - \mathbf{Z}_{fg} \mathbf{Y}^\Delta) & \text{vec}(\mathbf{X}_{gf} - \mathbf{Z}_{gf} \mathbf{Y}^\Delta) & \text{vec}(\mathbf{X}_{gg} - \mathbf{Z}_{gg} \mathbf{Y}^\Delta) \end{bmatrix} \quad (\text{F.43})$$

Equation (F.42) can be solved using either the singular value decomposition of \mathbf{W} $\text{svd}(\mathbf{W})$ or null of \mathbf{W} $\text{null}(\mathbf{W})$. Substituting the values of $\{\alpha_{ff}, \alpha_{fg}, \alpha_{gf}, \alpha_{gg}\}$ into equation (F.32). This approach obtains the i^{th} column of the right diagonalising SPE for the original system.

The left diagonalising transformation $\underline{\mathbf{T}}_{L0}$ can be calculated using equation (7.15).

$$\underline{\mathbf{T}}_{L0} = \begin{bmatrix} \mathbf{W}_{L0} & \mathbf{X}_{L0} \\ \mathbf{Y}_{L0} & \mathbf{Z}_{L0} \end{bmatrix} = \left(\begin{bmatrix} \mathbf{0} & \mathbf{K}_{D0} \\ \mathbf{K}_{D0} & \mathbf{D}_{D0} \end{bmatrix} \left(\begin{bmatrix} \mathbf{0} & \mathbf{K}_0 \\ \mathbf{K}_0 & \mathbf{D}_0 \end{bmatrix} \begin{bmatrix} \mathbf{W}_{R0} & \mathbf{X}_{R0} \\ \mathbf{Y}_{R0} & \mathbf{Z}_{R0} \end{bmatrix} \right)^{1-} \right)^T \quad (\text{F.44})$$

The above approach for a singular mass matrix can be simplified into steps:-

1. Transform the eigenvalue problem for the original system into a new system using Mobius transformations, equation (F.16) and divide the equation by $(qr - ps)$.
2. Find the right and left diagonalising SPEs $\{\underline{\mathbf{T}}_{RN}, \underline{\mathbf{T}}_{LN}\}$ for the new system.
3. Calculate the diagonalised system matrices $\{\mathbf{M}_{Do}, \mathbf{D}_{Do}, \mathbf{K}_{Do}\}$ using equation (F.19).
4. Extract the original system right and left diagonalising transformations $\{\underline{\mathbf{T}}_{Ro}, \underline{\mathbf{T}}_{Lo}\}$ from the new system using the homogenous coordinates definition as in equation (F.4).
 - Simplify the homogenous coordinates definition for the original system equation (F.4) into equation (F.14).
 - Applying Mobius transformations gives a relationship between the original system and the new system and vice versa. Substituting equation (F.17) into equation (F.14) yields equation (F.21).
 - For the new system, the homogenous coordinates can be simplified into equation (F.31).
 - The vectors $\{\mathbf{f}_{Roi}, \mathbf{g}_{Roi}\}$ in equation (F.21) and the vectors $\{\mathbf{f}_{RNi}, \mathbf{g}_{RNi}\}$ in equation (F.31) can be written as a linear combination with some unknown scalars $\{\alpha_{ff}, \alpha_{fg}, \alpha_{gf}, \alpha_{gg}\}$ as in equation (F.32). Substituting equation (F.32) into equation (F.21) leads to equation (F.34).
 - Comparing equation (F.34) with equation (F.31), we found that \mathbf{X} and \mathbf{Y} are equivalent for some invertible \mathbf{Z} . \mathbf{X} and \mathbf{Z} have each four components as in equations (F.40) and (F.41). These partitions can be calculated in turn by

setting the four unknown scalars $\{\alpha_{ff}, \alpha_{fg}, \alpha_{gf}, \alpha_{gg}\}$ to unity one at a time in equations (F.36) and (F.39), respectively.

- Substituting \mathbf{Z} in equation (F.41) back into equation (F.38) which is a function of $\{\alpha_{ff}, \alpha_{fg}, \alpha_{gf}, \alpha_{gg}\}$. The four unknown scalars $\{\alpha_{ff}, \alpha_{fg}, \alpha_{gf}, \alpha_{gg}\}$ are calculated from equations (F.38) and (F.40).

Bibliography

1. Carson, J.M., Powers, E.J., Stearman, R.O., Yi, E.J., and Ieee Comp Soc, I.C.S. *Applications of higher-order statistics to ground vibration testing of aircraft structures.* in *IEEE Signal Processing Workshop on Higher-Order Statistics*. 1999. Caesarea, Israel.
2. Abuazoum, L., *Experimental Study of Large Structures using Normal Mode Testing*, in *Mechanical Engineering*. 1996, Al-Fateh University, Faculty of Engineering: Tripoli-Libya.
3. Tayb, M.M., *Implementation of the dynamics modification theory for the prediction of the modal properties of structures*, in *Mechanical Engineering*. 1987, University of Nottingham: Nottingham.
4. Peeters, B., Hendricx, W., Debille, J., and Climent, H. *Modern Solutions for Ground Vibration Testing of Large Aircraft.* in *26th International Modal Analysis Conference*. 2008. Orlando, FL.
5. Peeters, B., Hendricx, W., Debille, J., and Climent, H., *Modern Solutions for Ground Vibration Testing of Large Aircraft*, in *Magazine Sound and Vibration* January 2009.
6. Balmes, E. and Soc Exptl, M. *GARTEUR group on ground vibration testing. Results from the test of a single structure by 12 laboratories in Europe.* in *15th International Modal Analysis Conference - Current Horizon for Structural Damage Detection (IMAC)*. 1997. Orlando, Fl.
7. McGowan, P.E., Angelucci, A.F., and Javeed, M., *Dynamic Test/Analysis Correlation using reduced analytical models*. NASA Technical Memorandum 107671. Langley Research Center Hampton, Virginia 23681, 1992.

8. Zaveri, K. and Phil, M., *Modal Analysis of Large Structures-Multiple Exciter Systems*. 1st ed. 1985: Brüel & Kjaer.
9. Caughey, T.K. and O'Kelly, M.E.J., *Classical normal modes in damped-linear dynamic systems*. ASME Journal of Applied Mechanics 1965. **32**(3): p. 583-588.
10. Ewins, D.J., *Modal Testing Theory, Practice and Application*. Second ed. 2000: Research Studies Press Ltd., Baldock, Hertfordshire, England.
11. Kehoe, M.W., *Aircraft ground vibration testing at NASA Ames-dryden Flight Research Facility*. 1987, NASA Ames Research Centre Dryden Flight Research Facility Edwards California, U.S.A. p. NASA TM-88272.
12. Brüel and Kjaer, *Structural testing using modal analysis*, Brüel & Kjaer BG 0112-11.
13. Wright, J., Cooper, J., and Desforges, M., *Normal-mode force appropriation - theory and application*. Mechanical Systems and Signal Processing, 1999. **13**(2): p. 217-240.
14. Cooper, J., Hamilton, M., and Wright, J., *Experimental Evaluation of Normal Mode Force Appropriation Methods using a rectangular plate*, in *10th International Modal Analysis Conference*. 1992. p. 1327-1333.
15. Wright, J., Holmes, P., Desforges, M., and Cooper, J., *Optimum Exciter Placement for Normal Mode Force Appropriation*. Proceedings of the Institution of Mechanical Engineers Part G-Journal of Aerospace Engineering, 1998. **212**(G5): p. 325-338.
16. Hickey, D., Worden, K., Wright, J.R., and Cooper, J.E., *Force Appropriation for Nonlinear Systems (FANS) Using Optimisation Methods*. Applied

Mechanics and Materials: Modern Practice in Stress and Vibration Analysis VI, 2006. **5-6**: p. 265-276.

17. Cooper, J., Hamilton, M., and Wright, J., *Eexperimental evaluation of various normal-mode force appropriation methods on a rectangular Perspex plate*. International Journal of Analytical and Experimental Modal Analysis, 1995. **10**(2): p. 118-130.
18. Friswell, M.I. and E.Mottershead, J., *Finite Element Model Updating in Structural Dynamics*. Solid Mechanics and its Applications, ed. G.M.L. GLADWELL. Vol. 38. 1995, Netherlands: Kluwer Academic Publishers.
19. Pickrel, C.R., *Airplane ground vibration testing - Nominal modal model correlation*. Sound and Vibration, 2002. **36**(11): p. 18-23.
20. Link, M. and friswell, M., *Generation of Validated Structural Dynamic Models-Results of a Benchmark Study Utilising the GARTEUR SM-AG19 Testbed*. Proceedings of ISMA, 2002. **III**: p. 1005-1018.
21. McGowan, P.E., Angelucci, A.F., and Javeed, M., *Dynamic Test/Analysis Correlation Using Reduced Analysis Modes*, in *NASA Technical Memorandum 107671*. 1992, NASA Langley Research Center, Hampton, Virginia 23681: Washington. p. 13.
22. Mottershead, J.E. and Friswell, M.I., *Model Updating in Structural Dynamics - A Survey*. Journal of Sound and Vibration, 1993. **167**(2): p. 347-375.
23. Garvey, S., Prells, U., and Friswell, M., *Modal correlation measures for general viscous-damped structures - A Clifford algebraic approach*, in *A Conference on Structural Dynamics*. 2001. p. 653-660.

24. Garvey, S.D. and Wood, M.G., *Integrated Systems Analysis* 2007, University of Nottingham: Nottingham.
25. Kehoe, M.W., Freudinger, L.C., Soc Exptl Mech, I.N.C., and Union, C. *Aircraft Ground Vibration Testing at the Nasa Dryden Flight Research Facility- 1993.* in *12th International Modal Analysis Conference.* 1994. Honolulu, Hi.
26. Kehoe, M.W. and Voracek, D.F. *Groung Vibration test results of a JETSTAR airplane using impulsive sine excitation.* in *International Modal Analysis Conference (IMAC) 7th.* 1989.
27. Razak, H.A., Choi, F.C., and Zingoni, A., *Damage Assessment of Corroded Reinforced Concrete Beams Using Modal Testing,* in *Structural Engineering, Mechanics and Computation.* 2001, Elsevier Science: Oxford. p. 1203-1215.
28. Friswell, M.I., Penny, J.E.T., and Wilson, D.A.L., *Using vibration data and statistical measures to locate damage in structures.* The International Journal of Analytical and Experimental Modal Analysis, 1994. **9**(4): p. 239-254.
29. Ge, M. and Lui, E.M., *Structural damage identification using system dynamic properties.* Computers & Structures, 2005. **83**(27): p. 2185-2196.
30. Hunt, D.L., *A Comparison of Methods for Aircraft Ground Vibration Testing,* in *3th International Modal Analysis Conference.* 1985. p. 131-137.
31. Olsen, N. *Excitation Functions for Structural Frequency Response Measurements.* in *Proceedings of the second International Modal Analysis Conference.* 1984. Orlando, Florida.
32. Kehoe, M.W., Cazier, F.W., Jr., and Ellison, J.F., *Ground Vibration Test of the Laminar Flow Control JetStar Airplane-1985.* 1985, NASA TM-86398.

33. Larue, P., Millet, M., and Piazzoli, G., *Pyrotechnic Bonkers for Structural tests in Flight*. 1974, Office National d'Etude et de Recherches Aerospatiales TP 1389.
34. Ma, F. and Caughey, T., *Analysis of Linear Nonconservative Vibrations*. Journal of Applied Mechanics-transactions of the ASME, 1995. **62**(3): p. 685-691.
35. Mitchell, L.D. *Complex Modes: A Review*. in *8th International modal analysis conference*. 1990.
36. Garvey, S.D., Penny, J.E.T., and Friswell, M.I., *The relationship between the real and imaginary parts of complex modes*. Journal of Sound and Vibration, 1998. **212**(1): p. 75-83.
37. Prells, U. and Friswell, M.I., *A measure of non proportional Damping*. Mechanical systems and signal processing, 2000. **14**(2): p. 125-137.
38. Kasai, T. and Link, M., *Identification of non-proportional modal damping matrix and real normal modes*. Mechanical Systems and Signal Processing, 2002. **16**(6): p. 921-934.
39. Adhiakri, S., *Classical Normal Modes in nonviscously Damped Linear Systems*. Journal of AIAA, 2001. **39**(5): p. 978-980.
40. Kreyszig, E., *Advanced engineering mathematics*. 8 ed. 1999, New York: John Wiley & Sons, Inc. 1156.
41. Thomson, W.T., *Theory of vibration with applications*. Second ed. 1981: Prentice-Hall, inc., Englewood Cliffs.

42. Garvey, S.D., Friswell, M.I., and Prells, U.a., *Coordinate Transformations for Second Order Systems, Part I: General Transformations*. Journal of Sound and Vibration, 2002. **258**(5): p. 885-909.
43. Garvey, S.D., Chen, Z., Friswell, M.I., and Prells, U. *Model reduction using structure-preserving transformations*. in *International Modal Analysis Conference IMAC, XXI*. 2003. Kissimmee, Florida.
44. Garvey, S.D., Friswell, M.I., and Prells, U.a., *Coordinate Transformations for Second Order Systems, Part II: Elementary Structure-Preserving Transformations*. Journal of Sound and Vibration, 2002. **258**(5): p. 911-930.
45. Chu, M.T. and Buono, N.D., *Total decoupling of general quadratic pencils, Part I: Theory*. Journal of Sound and Vibration, 2008. **309**(1-2): p. 96-111.
46. Chu, M.T. and Buono, N.D., *Total Decoupling of General Quadratic Pencils Part II: Structure preserving isospectral flows*. Journal of Sound and Vibration, 2008. **309**(1-2): p. 112-128.
47. Garvey, S., Prells, U., and Friswell, M. *Diagonalising coordinate transformations for general viscous-damped systems*. in *Proc. 19th Intl. Modal analysis conf*. 2001. Orlando, Florida.
48. Lancaster, P., *Symmetric transformations of the companion matrix*. NABLA: Bulletin of the Malayan Math. Soc., 1961. **8**: p. 146-148.
49. Friswell, M.I. and Penny, J.E.T., *Updating model parameters from frequency domain data via reduced order models*. Mechanical Systems and Signal Processing, 1990. **4**(5): p. 377-391.

50. O'Callahan, J., Avitabile, P., and Riemer, R. *System Equivalent Reduction Expansion Process (SEREP)*. in *International modal analysis conference, 7th*. 1989. Las Vegas.
51. O'Callahan, J. *A Procedure for an Improved Reduced System (IRS) Model*. in *International modal analysis conference 7th*. 1989. Las Vegas.
52. Guyan, R.J., *Reduction of Stiffness and Mass Matrices*. AIAA Journal, 1965. **3**(2): p. 380.
53. Irons, B., *Structural Eigenvalue Problems: Elimination of Unwanted Variables*. AIAA Journal, 1965. **3**(5): p. 961-962.
54. Anderson, R.G., Irons, B.M., and Zienkiewicz, O.C., *Vibration and Stability of Plates using Finite Elements*. International Journal of solids and structures, 1986. **4**: p. 1031-1055.
55. Wright, G.C. and Miles, G.A., *An Economical Method for Determining the Smallest Eigenvalues of Large Linear Systems*. International Journal for Numerical Methods in Engineering, 1971. **3**: p. 25-33.
56. Bathe, K.J. and Wilson, E.L., *Solution Method for Eigenvalues Problems in Structural Dynamics*. International Journal for Numerical Methods in Engineering, 1973. **6**(2): p. 213-226.
57. Jennings, A., *Mass Condensation and Simultaneous Iteration for Vibration Problems*. International Journal for Numerical Methods in Engineering, 1973. **6**(4): p. 543-552.
58. Kidder, R.L., *Reduction of Structural Frequency Equations*. AIAA Journal, 1973. **11**: p. 892.

59. Kidder, R.L., *Reply by Author to A. H. Flax*. AIAA Journal, 1975. **13**: p. 702-703.
60. Flax, A.H., *Comment on Reduction of Structural Frequency Equations*. AIAA Journal, 1975. **13**: p. 701-702.
61. Hensell, R.D. and Org, J.H., *Automatic Masters for Eigenvalue Economization*. Earthquake Engineering and Structural Dynamic, 1975. **3**: p. 375-383.
62. Irons, B.M., *Discussion on Dynamic Reduction of Structural models*. Journal of the Structural Division ASCE, 1981. **107**: p. 1023-1024.
63. Flax, A.H., *Discussion on Dynamic Reduction of Structural Models*. Journal of the Structural Division ASCE, 1981. **107**: p. 1393-1394.
64. Mario, P., *Dynamic Condensation*. AIAA Journal, 1984. **22**(5): p. 724-727.
65. Miller, C.A., *Dynamic Reduction of structural Models*. Journal of the Structural division 1980. **ST10**: p. 2097-2108.
66. Friswell, M.I., Garvey, S.D., and Penny, J.E.T., *Model reduction using dynamic and iterated IRS techniques*. Journal of Sound and Vibration, 1995. **186**(2): p. 311-323.
67. Gordis, J.H. *An analysis of the improved reduced system (IRS) model reduction procrdure*. in *Proceedings of the 10th International Modal Analysis Conference*. 1992. San Diego, California
68. Friswell, M.I., Garvey, S.D., and Penny, J.E.T., *The Convergence of the Iterated IRS Method*. Journal of Sound and Vibration, 1998. **211**(1): p. 123-132.

69. Friswell, M.I., Garvey, S.D., and Penny, J.E.T. *Using iterated IRS model reduction techniques to calculate eigensolutions.* in *15th International Modal Analysis Conference*. 1997. Orlando, Florida.
70. Qu, Z.Q. and Fu, Z.-F., *New Structural Dynamic Condensation Method for Finite Element Models.* AIAA Journal, 1998. **36**: p. 1320-1324.
71. Qu, Z.Q., Selvam, R.P., and Jung, Y., *Model condensation for non-classically damped systems-part II: Iterative schemes for dynamic condensation.* Mechanical Systems and Signal Processing, 2003. **17**(5): p. 1017-1032.
72. Qu, Z.Q., Jung, Y., and Selvam, R.P., *Model condensation for non-classically damped systems-part I: Static condensation.* Mechanical Systems and Signal Processing, 2003. **17**(5): p. 1003-1016.
73. Qu, Z.Q. and Chang, W., *Dynamic Condensation Method for Viscous Damped Vibration Systems in Engineering.* Engineering Structures, 2000. **22**(11): p. 1426-1432.
74. Garvey, S.D., Frisweel, M.I., and Penny, J.E.T., *A Clifford Algebraic Perspective on Second Order Linear Systems.* AIAA Journal of Control and Dynamics, 2000. **24**(1): p. 35-45.
75. Skelton, R.E., *Dynamic Systems Control: Linear Systems Analysis and Synthesis.* 1988: John Wiley & Sons.
76. Moore, B.C., *Principal Components Analysis in Linear Systems: Controllability, Observability and Model Reduction.* IEEE Trans. Automatic Control, 1981. **26**(1): p. 17-31.

77. Friswell, M.I., Penny, J.E.T., and Garvey, S.D., *The Application of the IRS and Balanced Realization Methods to obtain reduced models of structures with local non-linears*. Journal of Sound and Vibration, 1996. **196**(4): p. 453-468.
78. Hurty, W.C., *Dynamic analysis of structural systems using component modes*. AIAA Journal, 1965. **3**(4): p. 678-685.
79. Craig, R.R. *Coupling of Substructures for Dynamic Analysis: An overview*. in *Structures, Structural Dynamics, and Materials Conference, 41st AIAA/ASME/ASCE/AHS/ASC 2000*. Atlanta: AIAA-2000-1573.
80. Craig, R.R. and Bampton, M.C., *Coupling of Substructures for Dynamic Analysis*. AIAA Journal, 1968. **6**(7): p. 1313-1319.
81. Asher, G.W., *A method of Normal Mode Excitation using admittance measurements*. Proceedings of the 6th National Specialists Meeting on Dynamic and Aeroelasticity. Journal of Aeronautical sciences, 1958. **6**: p. 56-96.
82. Alexiou, K. and Wright, J.R., *Comparison of some Multi-point Force Appropriation Methods*. Modal Analysis, 1993. **8**(2): p. 119-136.
83. Traill-Nash, R.W., *Some Theoretical Aspects of Resonance Testing and Proposals for a Technique Combining Experiment and Computation*, in *Structures and Materials Report 280, Dept. of Supply, Australian Defence Scientific Service, Aeronautical Research Laboratories*. 1961.
84. Juang, J.N. and Wright, J.R., *A multi-point force appropriation method based upon a singular value decomposition approach*. Journal of vibration and acoustics, 1991. **113**: p. 176-181.

85. Allemang, R.J. and Brown, D., *A correlation coefficient for modal vector analysis*, in *1st International Modal Analysis Conference*. 1982, SEM: Orlando, Florida. p. 110-116.
86. Avitabile, P., O'Callahan, J., and Milani, J., *Model correlation and orthogonality criteria*, in *6th International Modal Analysis Conference*. 1988: Orlando, Florida. p. 1039-1047.
87. Morales, C.A., *Comments on the MAC and the NCO, and a linear modal correlation coefficient*. Journal of Sound and Vibration, 2005. **282**: p. 529-537.
88. Allemang, R.J., *The modal assurance criterion - Twenty years of use and abuse*. Journal of sound and vibration 2003. **37**(8): p. 14-24.
89. Garvey, S.D., Prells, U., and Friswell, M.I., *Quantifying the Correlation between Measures and Computed Mode Shapes*. Journal of Vibration and Control, 1996. **2**(2): p. 123-144.
90. Schedlinski, C., Link, M., and Schonrock, A., *Application of Computational Model Updating to Aeroengine Components*, in *Proc. of ISMA 23*. 1998: Leuven, Belgium.
91. Baruch, M. and Bar-Itzhack, I.Y., *Optimal weighted orthogonalization of measured modes*. AIAA Journal, 1978. **16**(4): p. 346-351.
92. Berman, A., *Comment on Optimal Weighted Orthogonalization of Measured Modes*. AIAA Journal, 1979. **17**(8): p. 927-928.
93. Barman, M. and Bar-Itzhack, I.Y., *Reply by Authors to a Berman AIAA Journal*, 1979. **17**: p. 928.

94. Baruch, M., *Optimal Correction of Mass and Stiffness Matrices Using Measured Modes*. AIAA Journal, 1982. **20**(11): p. 1623-1626.
95. O'Callahan, J.C. and Leung, R.K., *Optimization of Mass and Stiffness Matrices Using A Generalized Inverse Technique on the Measured Modes*, in *3th International Modal Analysis Conference*. 1985. p. 75-79.
96. Goge, D., *Automatic updating of large aircraft models using experimental data from ground vibration testing*. Aerospace Science and Technology, 2003. **7**(1): p. 33-45.
97. Fox, R.L. and Kapoor, M.P., *Rate of change of eigenvalues and eigenvectors*. AIAA Journal, 1968. **6**(12): p. 2426-2429.
98. Rogers, L.C., *Derivatives of eigenvalues and eigenvectors*. AIAA Journal, 1970. **8**(5): p. 943-944.
99. Rudisill, C.S. and Bhatia, K.G., *Optimization of complex structures to satisfy flutter requirements*. AIAA Journal 1971. **9**(8): p. 1487-1491.
100. Rudisill, C.S. and Bhatia, K.G., *Second derivatives of Flutter velocity and optimization of aircraft structures*. AIAA Journal 1972. **10**(12): p. 1569-1572.
101. Plaut, R.H. and Huseyin, K., *Derivative of eigenvalues and eigenvectors in non-self-adjoint systems*. AIAA Journal, 1973. **11**(2): p. 250-251.
102. Garg, S., *Derivatives of eigensolutions for a general matrix*. AIAA Journal, 1973. **11**(8): p. 1191-1194.
103. Rudisill, C.S., *Derivatives of eigenvalues and eigenvectors for a general matrix*. AIAA Journal, 1974. **12**(5): p. 721-722.

104. Nelson, R.B., *Simplified calculation of eigenvector derivatives*. AIAA Journal, 1978. **14**(9): p. 1201-1205.
105. Allen, J.J. and Martinez, D.R., *Techniques for Implementing Structural Identification using Test Data*. AIAA Journal, 1991. **29**(11): p. 1937-1944.
106. Brughmans, M., Leuridan, J., Hrycko, G., and Wyzykowski, J., *Application of FEM Model Correction and Updating Techniques on an Aircraft using test data of a Ground Vibration Survey*, in *15th International Modal Analysis Seminar*. 1990: K. U. Leuven. p. 589-606.
107. Becker, A.A., *Finite Element Analysis for Engineers*. Course Lecture Notes, March, 2006.
108. Pepper, D.W. and Heinrich, J.C., *The finite element method basic concepts and applications*. 2nd ed. 2006, London: CRC Press Taylor & Francis Group. 312.
109. Huebner, K.H., Thornton, E.A., and Byrom, T.G., *The finite element method for engineers*. 3rd ed. 1995, New York: John Wiley and Sons Inc. 627.
110. Huebner, K.H., Thornton, E.A., and Byrom, T.G., *The Finite Element Method for Engineers*. 3rd ed. 1995, New York, United States of America: John Wiley & Sons, Inc.
111. Bathe, K.-J., *Finite Element Procedures*. 1996, New Jersey: Prentice-Hall Inc. Simon & Schuster / A Viacom Company Upper Saddle River. 1037.
112. Gere, J.M. and Goodno, B.J., *Mechanics of Materials*. (SI Edition) 7 ed. 2009, Stamford, CT: Cengage Learning. 1002.

113. Zhiqiang, H., Yaong, C., and Zhongfang, T., *Modal Parameter Identification from Multi-point Measured Data*, in *3th International Modal Analysis Conference*. 1985. p. 138-144.
114. Kyong, B.L., Juang, J.-N., and Ghaemmaghami, P., *Eigenvector Derivatives of Repeated Eigenvalues using Singular Value Decomposition*. *Journal of Guidance Control and Dynamics*, 1989. **12**(2): p. 282-283.
115. Ojalvo, I., *Efficient computation of modal sensitivities for systems with repeated eigenvalues*. AIAA, 1988. **26**: p. 361-366.
116. Mills-Curran, W., *Calculation of eigenvector derivatives for structures with repeated eigenvalues*. AIAA, 1988. **26**: p. 867-871.
117. Dailey, R., *Eigenvector derivatives with repeated eigenvalues*. AIAA, 1989. **27**: p. 486-491.
118. Mills-Curran, W., *Comment on eigenvector derivatives with repeated eigenvalues*. AIAA, 1990. **28**: p. 1846.
119. Baisheng, W., Zhonghai, X., and Zhengguang, L., *Anote on computing eigenvector derivatives with distinct and repeated eigenvalues*. *Communications in numerical methods in engineering*, 2007. **23**: p. 241-251.
120. Mackey, D.S., *Structured Linearizations for Matrix Polynomials*, in *School of mathematics*. 2006, University of Manchester: Manchester.
121. Abuazoum, L. and Garvey, S.D. *Eigenvalue and Eigenvector Derivatives from the Perspective of Diagonalising Structure-Preserving*. in *Proceedings of XXVI International Modal Analysis Conference*. 2008. Orlando, Florida, USA.

122. Chandruptla, T.R. and Constans, E., *Technical Note: The use of Householder reflection in place of Givens rotation in matrix decomposition*. Advanced in Engineering Software, 2001. **32**: p. 673-674.
123. Golub, G. and Loan, C.V., *Matrix computations*, ed. second. 1989: Baltimore: Johns Hopkins university press.
124. Garvey, S.D., Prells, U., Friswell, M.I., and Chen, Z., *General isospectral flows for linear dynamic systems*. Linear Algebra and its Applications, 2004. **385**: p. 335-368.
125. Cardani, C. and Mantegazza, P., *Calculation of eigenvalue and eigenvector derivatives for algebraic Flutter and divergence eigenproblems*. AIAA Journal, 1979. **17**(6): p. 408-412.
126. Adhikari, S., *Rate of change of eigenvalues and eigenvectors in damped dynamic system*. AIAA Journal, 1999. **37**(11): p. 1452-1458.
127. Friswell, M.I. and Adhikari, S., *Derivatives of complex eigenvectors using Nelson's method*. AIAA Journal, 2000. **38**(12): p. 2355-2357.
128. Adhikari, S. and Friswell, M.I., *Eigenderivative analysis of asymmetric non-conservative systems*. International journal for numerical methods in engineering, 2001. **51**: p. 709-733.
129. Meerbergen, K. and Roose, D., *Matrix transformations for computing rightmost eigenvalues of large sparse non-symmetric eigenvalue problems*. Journal of Numerical Analysis, 1996. **16**(3): p. 297-346.
130. Kochina, P.Y. and Shishorina, O.I., *Möbius transformations and equations of empirical curves*. Journal of Applied Mathematics and Mechanics, 1993. **57**(1): p. 215-219.

131. Avitabile, P. and O'Callahan, J.C. *Model correlation and orthogonality criteria.* in *Proceedings of the 6th Modal Analysis Conference.* 1988. Orlando, Florida.
132. Pechinsky, F., Avitabile, P., and Foster, T. *Understanding the effects of DoF contribution to orthogonality checks.* in *Proceedings of the 11th Modal Analysis Conference.* 1993. Orlando, Florida.
133. Flanigan, C.C., Freymiller, J.E., Soc Exptl Mech, I.N.C., and Union, C. *Using Dynamic Expansion to Improve Test-Analysis Correlation.* in *12th International Modal Analysis Conference.* 1994. Honolulu, Hi.
134. Zhang, Z.Y., *A development of modal expansion method for eigensensitivity analysis of a defective matrix.* Applied Mathematics and Computation, 2007. **188**(2): p. 1995-2019.
135. Abuazoum, L.A. and Garvey, S.D., *Eigenvalue and eigenvector derivatives of second-order systems using structure-preserving equivalences.* Journal of Sound and Vibration, 2009. **323**: p. 1061-1076.
136. Adhikari, S., *Calculation of derivative of complex modes using classical normal modes.* Computers & Structures, 2000. **77**(6): p. 625-633.
137. Schedlinski, C., Wagner, F., Bonhnert, K., Frappier, J., Irrgang, A., Lehmann, R., and Muller, A. *Experimental Modal Analysis and Computational Model Updating of a car Body in White.* in *ISMA.* 2004. Leuven, Belgium.
138. Michael, L., Stefan, S., and Matthias, W., *Identification of structural property degradations by computational model updating.* Engineering Materials, 2007. **34**(7): p. 19-34.
139. *Dynamic Design Solutions NV (DDS), FEMTools - Theoretical Manual.* 2004: Leuven, Belgium.

140. Horn, R.A. and Jonson, C.R., *Matrix Analysis*. 1992: Cambridge University Press, Cambridge.
141. Gohberg, I., Lancaster, P., and Rodman, L., *Matrix Polynomials*. Academic Press. 1982, New York.
142. Lancaster, P., *Isospectral Vibrating Systems. Part 1 : The Spectral Method*. Linear Algebra and its Applications, 2005. **409**: p. 51-69.
143. Takewaki, I. and Uetani, K., *Inverse component-mode synthesis method for damped large structural systems*. Computers & Structures, 2000. **78**(1-3): p. 415-423.
144. Rubin, S., *Improved component moderepresentation for structural dynamic analysis*. AIAA Journal, 1975. **13**(8): p. 995-1006.
145. Houlston, P.R., *Active Vibration Control of Rotating Machines*, in *Mechanical Engineering*. 2007, University of Nottingham: Nottingham. p. 213.